FURTHER MATHEMATICS

Paper 9231/11
Paper 1

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they use all the information given and answer all aspects in adequate depth. They should take note of where exact answers are required and show the method being used.

Candidates should ensure that any sketch graphs are fully labelled and carefully drawn to show significant points and behaviour at limits.

Both algebra and arithmetic can often be simplified by the use of common factors.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations.

Comments on specific questions

Question 1

(a) The most effective solutions were those where candidates started by writing the coefficients of the cubic equation in terms of the roots α, α, γ. This gave three equations in two unknowns and eliminating α, γ led to the required result. Many candidates omitted $2\alpha\gamma + \alpha^2 = 0$ and were unable to proceed. Those candidates who attempted to use $S_3 + bS_2 + 3d = 0$ often made little progress because they, too, omitted the information given by the zero coefficient of x.

(b) There were many good solutions to this part of the question. Candidates were able to use the result from **(a)** with the familiar formula for finding the sum of the squares of the roots to find the required values. The best solutions mentioned the condition $d \neq 0$ to exclude the possibility $b = 0$.

Question 2

The structure of a proof by induction was clearly well understood and many solutions earned high marks. Those candidates who started with the expression for the kth term being divisible by 48 and moved towards the expression for the $(k + 1)$th term being divisible, were usually correct in their algebra. Those who showed that the difference between the two terms is divisible by 48 needed to remember to link that result to their hypothesis in order to complete the step. Almost all candidates stated that the result holds and most remembered to state 'for all positive integers n'.

Question 3

(a) Most candidates could correctly apply the method of differences to find the sum of $(2r+1)^2-(2r-1)^2$. To make progress with the question this expression also needed to be expanded and simplified so that both ways of reaching the sum could be equated. There were many accurate proofs, with the most common error being to forget to sum the second term when finding $\sum_{r=1}^{n}(24r^2+2)$.

(b) Candidates needed to recognise that the odd and even terms of the series needed to be treated separately. The best solutions showed this and dealt efficiently with the required algebra.

(c) The method for finding a sum to infinity was accurately applied.

Question 4

This question was of a familiar type and was answered well.

As with all vector questions, mistakes in arithmetic and with signs can impact on the level of accuracy. Candidates should develop the habit of checking carefully.

It is also important to show which vectors are being used so that the method can be determined even if there is a numerical error.

(a) Almost all candidates used the cross-product of correct vectors to find the normal to the plane and substituted to find the Cartesian equation. Errors were mainly numerical.

(b) There were many accurate calculations of the angle between normal and line and the strongest candidates remembered to subtract from 90° to find the required answer.

(c) Most candidates realised that they should express the position vector of the foot of the perpendicular in terms of a parameter. A common error was to use the vector **j** + **k** instead of the normal to the plane. Most candidates gave their answer in terms of a position vector as required.

Question 5

(a) The transformations were usually correctly named and many candidates gave the correct order.

(b) This question asked for the inverse of matrix **M**. Most candidates found **M** but did not go on to find its inverse.

(c) This topic was well understood, and the majority of candidates showed clearly that they were using a correct method. The most efficient solutions cancelled out the factor of $\frac{\sqrt{2}}{2}$ as soon as possible in order to simplify the algebra.

Question 6

(a) The majority of candidates gave a completely correct proof of the result.

(b) There were many good sketches, showing a single loop with symmetry in the initial line and correct shape when $\theta = 0$ and $\theta = \pm\frac{1}{4}\pi$.

(c) Most answers to this part were correct.

(d) Some candidates attempted to maximise *x* or *r*, but the majority of attempts worked with *y* as required. There were many correct differentiations, the most common errors being numerical. Candidates clearly knew how to use trigonometric formulae to solve the equation to find θ. However, a number used their value of θ to find *r* instead of $y = r\sin\theta$.

Question 7

This question was done well, with many completely correct solutions.

(a) The vertical asymptote was correctly stated by most candidates. The usual method for finding the oblique asymptote was long division and most candidates continued the process to obtain the constant term in the equation of the line.

(b) The differentiation was done well and the coordinates of both maximum and minimum were found accurately.

(c) There were many careful sketches, with good shape and correct labelling.

(d) The transformation of the graph was properly done. Both equations to find the critical points were almost always solved correctly. The final inequalities were usually correct.

FURTHER MATHEMATICS

Paper 9231/12
Paper 1

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they use all the information given and answer all aspects in adequate depth. They should take note of where exact answers are required and show the method being used.

Candidates should ensure that any sketch graphs are fully labelled and carefully drawn to show significant points and behaviour at limits.

Both algebra and arithmetic can often be simplified by the use of common factors.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations. It seemed that almost all were able to complete the paper in the time allowed.

Comments on specific questions

Question 1

This question was generally answered well.

(a) Almost all candidates chose the appropriate formulae and simplified their answer to an acceptable form.

(b) The partial fractions were usually correct. Many responses were carefully set out to demonstrate clear application of the method of differences. Candidates need to remember that to demonstrate the pattern of cancellation enough terms must be shown. In this case, three consecutive values of r were needed to show the pattern. Those who gave only the last term often omitted one term of the answer.

(c) Most responses followed correctly from the answer for **(b)**.

Question 2

Some candidates became confused by which roots belonged to which equation. Others avoided this by using different letters for the roots.

(a) There were many completely correct solutions for this part of the question. The usual method was to make the substitution $x = y^{-\frac{1}{2}}$. There were some numerical errors but most candidates realised that they needed to isolate the terms involving fractional powers before squaring to eliminate them. Others expressed the given equation in terms of x^2 and then replaced this by y^{-1}. Only a few candidates gave an answer which was not a quartic equation. Several candidates did not answer the last part.

(b) Those candidates who noticed the relationship between $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$ from **(a)** and $\beta^2\gamma^2\delta^2+\alpha^2\gamma^2\delta^2+\alpha^2\beta^2\delta^2+\alpha^2\beta^2\gamma^2$ produced an elegant solution. Some candidates divided by 36 rather than multiplied. The other successful method was to use a substitution to find the equation with roots α^2 etc. and write down the result from its coefficients.

(c) This part was very well answered.

Question 3

(a) The terms stretch and shear were well known. The idea that the right hand matrix should be applied first was often forgotten.

(b) Many candidates thought that this question expected them to write down the inverse of matrix **M**. Instead they needed to write down the inverse of the shear matrix and the inverse of the stretch matrix, and to write them in the correct order. There were a number of incorrect inverse matrices.

(c) This question asked specifically for the invariant points of matrix **M**. A large proportion of candidates used a method to find the gradient of the invariant line of the matrix – the points of an invariant line are not, in general, invariant points. Those candidates who wrote an equation using the invariance of the point found the required line efficiently and accurately.

(d) The relationship between the determinant of the transformation and area was clearly well known. Here, the information on the triangle areas told candidates that |det**M**| = 1. Many presumed that this meant det**M** = 1, and hence that k = 1, contradicting the information given at the beginning of the question.

Question 4

All candidates seemed to have a good understanding of the structure of a proof by induction. Most candidates demonstrated that the formula is correct for n = 1 by using the product rule and stated that the base case is true. The assumption for a single value of k was usually clearly expressed. Many stronger candidates then wrote down the target expression. The inductive step needed two differentiations using the product rule. The strongest candidates performed these separately, using the $f''(x) = f(x)$ to tidy up and reach the target expression. Almost all candidates stated that the result holds and most remembered to state 'for all positive integers n'. The most common errors were in the differentiation.

Question 5

(a) Most candidates produced a sketch on the correct domain and with increasing r. Only the strongest candidates showed the correct curvature. When writing down the polar coordinates of the extreme points, many candidates gave the values separately or as (θ, r). A considerable number omitted the multiplier a from their r values.

(b) Stronger candidates wrote down $y = 2a\sin\left(\frac{\pi}{4}\right)$ by considering their diagram. There were also attempts to differentiate the general expression for y in order to maximise it. Many others found the greatest value of r or of x.

(c) Almost all candidates could correctly write down the required integral. The most efficient solutions simply used $\sec^2\theta = 1 + \tan^2\theta$ and wrote the integrand as $\sec^2\theta + \tan^2\theta\sec^2\theta$. This could be integrated quickly. Other successful methods included integration by parts and use of a reduction formula. The majority of candidates made little progress in performing the integration. Not all showed their method clearly - answers supported by careful working show a candidate's understanding and are more likely to lead to the correct result.

(d) There were some neat and efficient representations of the curve in Cartesian form. Often candidates became confused with the algebra, and those who found themselves multiplying out $(x^2 + y^2)^2$ rarely made any further progress. It is always helpful to make sure that any common factors are removed to keep the expressions as simple as possible. Those candidates who showed that $x^2 = ar$ usually reached the correct equation for *y,* but several forgot to square *a* at some point.

Question 6

As with all vector questions, mistakes in arithmetic and with signs resulted in a loss of accuracy. Candidates should develop the habit of checking carefully.

There was quite frequent confusion between a position vector, a vector joining two points and a direction vector. Careful labelling can help.

It was also important to show which vectors were being used so that the method could be determined even if there was a numerical error.

(a) Many candidates used the most efficient method of first finding the direction of the common perpendicular. They then found a vector joining a point of l_1 to a point of l_2 and used the scalar product to find its projection on to the common perpendicular.

A number of candidates then felt the need to effectively repeat the working. They used the other popular method of finding the parameters corresponding to the points *P* and *Q* and hence the length *PQ*. The parameters involved fractions and there was great opportunity for error in the arithmetic.

(b) (i) The question asked for an equation and this required *r* = at the beginning. The majority of candidates understood that the directions $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$ were needed. The most common error was to use $\begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix}$ instead of the direction of the common perpendicular.

(ii) The usual method was to take the cross product of two vectors. Most candidates used the direction of l_2 and what they believed to be the common perpendicular from **(a)**. Efficient solutions used the directions in their simplest form rather than the vector *PQ*. There were a number of errors in evaluating the cross product and even some candidates who used position vectors.

(c) There were many completely correct solutions for this part. The two normals were found and the formula for the angle between them used accurately. Candidates who had made numerical and other errors earlier in the question did not always make it clear which vectors they were using and it was not possible to identify that they were using an appropriate method.

Question 7

This question was, in general, answered well.

(a) The procedure for finding the asymptotes was well understood. Only a few candidates missed out the constant term of the oblique asymptote.

(b) The differentiation was performed accurately, the equations solved and the exact coordinates of the stationary points given clearly.

(c) There were some very good sketch graphs drawn. Asymptotes were labelled with their equations and the curve often approached them steadily and closely. A few candidates omitted the coordinates of the intersection of curve and axes. The two branches of the curve were usually correct.

(d) The idea of reflecting the graph in the *x*-axis was well understood, the most common error was to leave the part of the graph between $x = 0$ and $x = 1$ below the axis.

The two cases of the function equal to $+6$ and -6 were considered and critical values found accurately. Those candidates who worked with equations rather than inequalities at this point were less likely to become confused when giving the final answer as two ranges of value for *x*. However, a number of candidates wrote ‘ $-2 < x < -3$’. It is always worth checking that inequalities make sense.

FURTHER MATHEMATICS

Paper 9231/13
Paper 1

Key messages

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General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations.

Comments on specific questions

Question 1

(a) The most effective solutions were those where candidates started by writing the coefficients of the cubic equation in terms of the roots α, α, γ. This gave three equations in two unknowns and eliminating α, γ led to the required result. Many candidates omitted $2\alpha\gamma + \alpha^2 = 0$ and were unable to proceed. Those candidates who attempted to use $S_3 + bS_2 + 3d = 0$ often made little progress because they, too, omitted the information given by the zero coefficient of x.

(b) There were many good solutions to this part of the question. Candidates were able to use the result from **(a)** with the familiar formula for finding the sum of the squares of the roots to find the required values. The best solutions mentioned the condition $d \neq 0$ to exclude the possibility $b = 0$.

Question 2

The structure of a proof by induction was clearly well understood and many solutions earned high marks. Those candidates who started with the expression for the kth term being divisible by 48 and moved towards the expression for the $(k + 1)$th term being divisible, were usually correct in their algebra. Those who showed that the difference between the two terms is divisible by 48 needed to remember to link that result to their hypothesis in order to complete the step. Almost all candidates stated that the result holds and most remembered to state 'for all positive integers n'.

Question 3

(a) Most candidates could correctly apply the method of differences to find the sum of $(2r+1)^2-(2r-1)^2$. To make progress with the question this expression also needed to be expanded and simplified so that both ways of reaching the sum could be equated. There were many accurate proofs, with the most common error being to forget to sum the second term when finding $\sum_{r=1}^{n}(24r^2+2)$.

(b) Candidates needed to recognise that the odd and even terms of the series needed to be treated separately. The best solutions showed this and dealt efficiently with the required algebra.

(c) The method for finding a sum to infinity was accurately applied.

Question 4

This question was of a familiar type and was answered well.

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It is also important to show which vectors are being used so that the method can be determined even if there is a numerical error.

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(c) Most candidates realised that they should express the position vector of the foot of the perpendicular in terms of a parameter. A common error was to use the vector **j** + **k** instead of the normal to the plane. Most candidates gave their answer in terms of a position vector as required.

Question 5

(a) The transformations were usually correctly named and many candidates gave the correct order.

(b) This question asked for the inverse of matrix **M**. Most candidates found **M** but did not go on to find its inverse.

(c) This topic was well understood, and the majority of candidates showed clearly that they were using a correct method. The most efficient solutions cancelled out the factor of $\frac{\sqrt{2}}{2}$ as soon as possible in order to simplify the algebra.

Question 6

(a) The majority of candidates gave a completely correct proof of the result.

(b) There were many good sketches, showing a single loop with symmetry in the initial line and correct shape when $\theta = 0$ and $\theta = \pm\frac{1}{4}\pi$.

(c) Most answers to this part were correct.

(d) Some candidates attempted to maximise *x* or *r*, but the majority of attempts worked with *y* as required. There were many correct differentiations, the most common errors being numerical. Candidates clearly knew how to use trigonometric formulae to solve the equation to find θ. However, a number used their value of θ to find *r* instead of $y = r\sin\theta$.

Question 7

This question was done well, with many completely correct solutions.

(a) The vertical asymptote was correctly stated by most candidates. The usual method for finding the oblique asymptote was long division and most candidates continued the process to obtain the constant term in the equation of the line.

(b) The differentiation was done well and the coordinates of both maximum and minimum were found accurately.

(c) There were many careful sketches, with good shape and correct labelling.

(d) The transformation of the graph was properly done. Both equations to find the critical points were almost always solved correctly. The final inequalities were usually correct.

FURTHER MATHEMATICS

Paper 9231/21
Paper 2

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.
- Candidates should make use of results derived or given in earlier parts of a question as appropriate.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. Sometimes candidates reached conclusions without justification, particularly where answers were given within the question. There were many responses of a high standard.

Comments on specific questions

Question 1

This part was very well done with most candidates accurately differentiating twice to find the Maclaurin's series, but a few candidates accepted an erroneous value of f'(0) without checking their work.

Question 2

(a) Almost all candidates found the determinant correctly.

(b) While it was common to see row operations or elimination of a variable, only stronger candidates gave a full geometric interpretation by emphasising that the three planes form a sheaf.

(c) Stronger candidates derived a clear contradiction and interpreted the situation correctly as a triangular prism.

Question 3

Most candidates accurately recalled the formula for the length of the curve with correct limits. Stronger candidates fully simplified $\sqrt{\dot{x}^2 + \dot{y}^2}$ before substituting into the formula, which caused fewer errors and enabled a clear path to the answer.

Question 4

(a) Almost all candidates worked from LHS to RHS, after writing cosh and sinh in terms of exponentials, to fully justify the given identity.

(b) It was common to see the chain rule applied correctly, with the majority of candidates multiplying by $\cosh x$ after using the formula for the derivative of $\tan^{-1} u$.

(c) Sketches were mostly accurate with the equation of the asymptote clearly stated as $y = 0$. A few candidates neglected to make their curve smooth at stationary point $(0,1)$.

(d) Most candidates formed a correct expression for the sum of the areas of the rectangles and integrated accurately to derive the given result.

(e) Most candidates correctly adapted their solution to **(e)** and gave a suitable upper bound.

Question 5

Almost all candidates knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the particular integral and some errors when substituting initial conditions. A few candidates gave expressions instead of equations as their answer.

Question 6

(a) This part of the question was done well, but a few candidates accepted zero eigenvectors without checking for errors in their working. A few candidates spent time finding $\det(\mathbf{A} - \lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix.

(b) Stronger candidates were able to maintain accuracy throughout their solution, both when manipulating the characteristic equation and when substituting $\mathbf{A}$ and making $\mathbf{A}^3$ the subject.

Question 7

(a) The majority of candidates used the formula for the sum of a geometric progression accurately.

(b) The majority of candidates applied de Moivre's theorem to fully justify the given result.

(c) Almost all candidates knew that de Moivre's theorem related the series to the geometric progression in **(a)**. Stronger candidates accurately took the imaginary part, after simplifying the numerator and denominator, which led to the given answer.

(d) Stronger candidates maintained accuracy when substituting $\theta = \frac{1}{3}\pi$ and $n = 6m$, fully simplifying their answer.

Question 8

(a) Almost all candidates used the given substitution and integrated to get $-\sqrt{1-(\theta-1)^2} + C$, but it was common to see the arbitrary constant omitted.

(b) The strongest candidates maintained accuracy and fully simplified the right hand side of the equation after multiplying by the integrating factor. Using the answer to **(a)** and integration by parts, the integral of the right hand side could be derived as $(\theta-1)\sin^{-1}(\theta-1) + \sqrt{1-(\theta-1)^2} + C$ and the initial conditions substituted to find the correct value of C.

FURTHER MATHEMATICS

Paper 9231/22
Paper 2

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.
- Candidates should make use of results derived or given in earlier parts of a question as appropriate.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. Sometimes candidates reached conclusions without justification, particularly where answers were given within the question. There were many responses of a high standard.

Comments on specific questions

Question 1

(a) This was very well done with most candidates accurately finding the determinant as $6k-12$ and setting it to be non-zero. The small number of candidates who instead solved the system of equations in terms of k were less successful.

(b) The strongest candidates gave a full geometric description, stating that the two parallel planes are not identical.

Question 2

(a) The majority of candidates found the first derivative correctly using implicit differentiation.

(b) The attempts to find the second derivative varied in length, with stronger candidates showing the required level of algebraic fluency when differentiating expressions such as yy' with respect to x and maintaining accuracy when substituting into their expression.

Question 3

(a) Most candidates accurately recalled the formula for surface area with correct limits. Stronger candidates fully simplified $\sqrt{1+y'^2}$ before substituting into the formula, which caused fewer errors and enabled a clear path to the answer.

(b) The majority of candidates used the expansion of e^x correctly. Deriving the Maclaurin's series by differentiating $\mathrm{e}^x+\frac{1}{4}\mathrm{e}^{-x}$ twice was also a successful approach.

Question 4

Most candidates used the correct method to find the integrating factor, but errors with constants were common and, after multiplying both sides of the equation by their integrating factor, most candidates were able to attempt to integrate the RHS. Stronger candidates maintained accuracy throughout, particularly when substituting in the initial conditions. Most candidates remembered to add an arbitrary constant after integrating the right-hand side, substituted the initial conditions and made x the subject.

Question 5

(a) Almost all candidates wrote down the four fourth roots of unity, often in exponential or trigonometric form.

(b) After expanding $(\cos\theta + \mathbf{i}\sin\theta)^4$ using the binomial expansion, most candidates grouped together terms contributing to the real part before applying the identity $\cos^2\theta + \sin^2\theta = 1$ to fully justify the given result. Alternatively, a few candidates worked from the right-hand side to the left-hand side, applying $2\cos\theta = z + z^{-1}$, and were usually successful too.

(c) Almost all candidates factorised correctly, linking with the previous part of the question. Stronger candidates then solved $\cos 4\theta = \pm\frac{1}{2}\sqrt{3}$, remembering to consider both signs, leading to exactly eight distinct real roots.

Question 6

Stronger candidates showed consideration of the sum of the areas of the rectangles to justify the left-hand side of the inequality. When dealing with the right-hand side, the most common approach was to complete the square and use the logarithmic form of the inverse of cosh to justify the given answer. A small number of candidates worked directly with the natural logarithm via inspection, and were usually successful too.

Question 7

(a) While it was common to see $\mathbf{Ae} = \lambda\mathbf{e}$ written, only the strongest candidates gave a correct algebraic argument by multiplying both sides of the equation by $\mathbf{A}^{-1}$ on the left-hand side.

(b) This part was done well with most candidates finding the cross product of two rows of $\mathbf{A} + \mathbf{I}$, rather than solving simultaneous equations.

(c) This part of the question was also done well with the majority of candidates finding the correct eigenvalues.

(d) Most candidates took reciprocals of their eigenvalues and matched them correctly to their eigenvectors, but a few candidates accepted zero eigenvectors without checking for errors in their working.

(e) Stronger candidates were able to maintain accuracy when manipulating the characteristic equation, by replacing λ with $\mathbf{A}$ and by multiplying both sides of their equation by $\mathbf{A}^{-1}$.

Question 8

(a) This part was done well with candidates finding the first two derivatives of y or, alternatively, u with respect to x and substituting to justify the given equation. Stronger candidates clearly worked from the given $y - x$ equation to the given $u - x$ equation, applying $\cosh^2 u - \sinh^2 u = 1$.

(b) Almost all candidates knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the constants and some problems with notation. A few candidates gave expressions instead of equations as their answer. More successful approaches involved substituting $y = \cosh u$ and making u the subject.

FURTHER MATHEMATICS

Paper 9231/23
Paper 2

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.
- Candidates should make use of results derived or given in earlier parts of a question as appropriate.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. Sometimes candidates reached conclusions without justification, particularly where answers were given within the question. There were many responses of a high standard.

Comments on specific questions

Question 1

This part was very well done with most candidates accurately differentiating twice to find the Maclaurin's series, but a few candidates accepted an erroneous value of f'(0) without checking their work.

Question 2

(a) Almost all candidates found the determinant correctly.

(b) While it was common to see row operations or elimination of a variable, only stronger candidates gave a full geometric interpretation by emphasising that the three planes form a sheaf.

(c) Stronger candidates derived a clear contradiction and interpreted the situation correctly as a triangular prism.

Question 3

Most candidates accurately recalled the formula for the length of the curve with correct limits. Stronger candidates fully simplified $\sqrt{\dot{x}^2 + \dot{y}^2}$ before substituting into the formula, which caused fewer errors and enabled a clear path to the answer.

Question 4

(a) Almost all candidates worked from LHS to RHS, after writing cosh and sinh in terms of exponentials, to fully justify the given identity.

(b) It was common to see the chain rule applied correctly, with the majority of candidates multiplying by $\cosh x$ after using the formula for the derivative of $\tan^{-1} u$.

(c) Sketches were mostly accurate with the equation of the asymptote clearly stated as $y = 0$. A few candidates neglected to make their curve smooth at stationary point $(0,1)$.

(d) Most candidates formed a correct expression for the sum of the areas of the rectangles and integrated accurately to derive the given result.

(e) Most candidates correctly adapted their solution to **(e)** and gave a suitable upper bound.

Question 5

Almost all candidates knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the particular integral and some errors when substituting initial conditions. A few candidates gave expressions instead of equations as their answer.

Question 6

(a) This part of the question was done well, but a few candidates accepted zero eigenvectors without checking for errors in their working. A few candidates spent time finding $\det(\mathbf{A}-\lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix.

(b) Stronger candidates were able to maintain accuracy throughout their solution, both when manipulating the characteristic equation and when substituting $\mathbf{A}$ and making $\mathbf{A}^3$ the subject.

Question 7

(a) The majority of candidates used the formula for the sum of a geometric progression accurately.

(b) The majority of candidates applied de Moivre's theorem to fully justify the given result.

(c) Almost all candidates knew that de Moivre's theorem related the series to the geometric progression in **(a)**. Stronger candidates accurately took the imaginary part, after simplifying the numerator and denominator, which led to the given answer.

(d) Stronger candidates maintained accuracy when substituting $\theta = \frac{1}{3}\pi$ and $n = 6m$, fully simplifying their answer.

Question 8

(a) Almost all candidates used the given substitution and integrated to get $-\sqrt{1-(\theta-1)^2}+C$, but it was common to see the arbitrary constant omitted.

(b) The strongest candidates maintained accuracy and fully simplified the right hand side of the equation after multiplying by the integrating factor. Using the answer to **(a)** and integration by parts, the integral of the right hand side could be derived as $(\theta-1)\sin^{-1}(\theta-1)+\sqrt{1-(\theta-1)^2}+C$ and the initial conditions substituted to find the correct value of C.

FURTHER MATHEMATICS

Paper 9231/31
Paper 3

Key messages

A diagram is often an invaluable tool in helping candidates to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates may draw their own diagram as well.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that there is no doubt that the offered solution is clear and complete. However, in all questions candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

Candidates should be encouraged to draw a suitable diagram or, in case a diagram is provided, to annotated it, as this helps understand the problem and model it correctly. For example, in **Question 3**, candidates who correctly annotated the diagram provided, realised that there were five forces involved and were able to determine the best point about which to work out the moments equation in **(a)**, as well as to correctly write the equations of forces in **(b)**.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's second law, e.g. to set up a differential equation or in questions involving collisions, they must ensure they explicitly mention the mass or masses involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra, and not to quote formulae, but rather to derive them from first principles, as in **Question 6a**.

Comments on specific questions

Question 1

Most candidates were able to work out the correct expression for the angular velocity. Stronger candidates then used the angular velocity to obtain the correct answer, without determining the period of revolution. A common mistake was to omit the multiplication by 60, therefore obtaining the number of revolutions per second.

Question 2

(a) Many candidates correctly identified expressions for the kinetic and elastic potential energy in the two situations, equated them and obtained the correct answers, often in a very elegant way. Some candidates attempted a variety of incorrect approaches (SUVAT, equilibrium of forces), which invariably led to an incorrect answer and rarely gained any credit. A common mistake was to omit the coefficient 2 in the denominator of the elastic potential energy, or to incorrectly simplify one of the terms of the elastic potential energy into mga or $\frac{1}{4}mga$ instead of $\frac{1}{2}mga$ or, respectively, $\frac{1}{8}mga$.

(b) This question proved more challenging than the previous one. Common mistakes were in the expression for the extension of the string, in the use of Hooke's formula (e.g., including a factor of 2 in the denominator, or adding an additional term $-mg$) or, as for **(a)** in the use of an incorrect approach, often SUVAT. More efficient approaches reached the solution in two well-presented steps.

Question 3

(a) Candidates who wrote the equation of the moments about point *A* often obtained the correct answer. Those who chose another point very rarely managed to gain credit, as their equations usually did not contain the correct number of terms. Stronger candidates annotated the diagram provided with an indication of the forces acting at the different points and stated explicitly the point about which they were taking the moments. Candidates are strongly encouraged to annotate the diagram, especially in situations where there are many forces acting at different points, as well as to explicitly state the point about which they are taking moments.

(b) This question proved challenging for most candidates who attempted to resolve forces in only one direction, vertical, horizontal, parallel, or perpendicular to the rod. Often, they did not include all components, and this would have been evident had they annotated the diagram. Stronger candidates realised that the best strategy was to resolve forces horizontally and vertically. They usually wrote the correct system of equations and obtained the correct answer, often showing strong algebraic manipulative skills.

Question 4

(a) Most candidates wrote the correct differential equation and separated the variables. However, only some of them used partial fractions to integrate the equations and only stronger candidates obtained the correct solution. These candidates almost always determined the correct value for the constant of integration from the boundary condition and then obtained the correct answer, showing mastery in the manipulation of logarithms.

(b) Even though the correct answer could be obtained without integrating the equation in **(a)**, very few candidates who did not do so were able to score the mark allocated to this part question. However, candidates who did well in **(a)**, usually had no problems in reaching the correct answer.

Question 5

(a) To obtain the correct answer for this question, candidates were required to perform three steps.

1. Use the equation for the conservation of energy at the endpoints.
2. Use Newton's second law at point.
3. Solve the resulting system of simultaneous equations to obtain the values of k and θ .

Most candidates realised this and attempted to write the equations, with various degrees of success. A common mistake was to invert the sign for the gravitational potential energy in the energy equation and/or the term due to gravity in Newton's second law, thus writing $\frac{1}{2}mkag + mga\cos\theta = \frac{1}{6}mag$ and/or, respectively, $\frac{11}{6}mg + mg\cos\theta = mkg$.

(b) Candidates used a variety of approaches to answer this question. They mostly involved one or more SUVAT equations, with a few candidates choosing the conservation of energy. Even though the majority of approaches were correct, only stronger candidates selected the SUVAT equation $v^2 = u^2 + 2as$, which quickly led to the correct answer. Some candidates opted for the more complicated solution of a system of two SUVAT equations involving time, e.g. $v = u + at$ and $s = ut + \frac{1}{2}at^2$, often obtaining an incorrect answer.

Question 6

(a) Many candidates found this question rather challenging. Most candidates wrote the momentum equation correctly, but only a few could then write the corresponding equation for Newton's elastic law with consistent signs for the velocities involved. Stronger candidates were able to write two correct equations and solved them flawlessly, often in an elegant manner.

(b) Most candidates understood that this question required them to calculate the initial and final energy and subtract the former from the latter to score the method mark. Many realised that the component of the velocity of particle B perpendicular to the lines of centre was $\frac{5}{8}u\sin(\alpha)$. The majority of stronger candidates who scored full marks in **(a)** scored full marks in this question too.

Question 7

(a) In this question candidates were asked to show that the total time of flight was equal to a given expression. Most of them realised that as the answer was given, they were expected to show the result from first principles and to carefully explain each step.

(b) This question also asked candidates to explain a given result. The key concept to express was that the vertical component of the velocity is not affected by the impact, as the wall is vertical. Many candidates were able to explain this, and some gave more elaborated explanations that showed a deep understanding of the underlying concepts.

(c) To obtain the correct answer for this question, candidates were required to perform three steps:

1. Calculate the time taken to hit the wall using the horizontal component of the velocity.
2. Calculate the time taken to hit the ground after hitting the wall, again using the horizontal component of the velocity, after the collision.
3. Add the two times, equate the sum to the expression provided in **(a)**, and solve the equation.

Many candidates were able to determine the horizontal component of the speed before and after the collision, but only the strongest used this information, along with the fact that the barrier was 15 metres from point O, to obtain the expressions for the times. Most of those who did, and who wrote the correct equation, had no difficulty solving the equation. Stronger candidates did this without evaluating $\cos(75°)$ and $\sin(75°)$ explicitly, but instead used the double angle formula.

FURTHER MATHEMATICS

Paper 9231/32
Paper 3 Further Mechanics

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram as well.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the offered solution is clear and complete. In all questions, candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

In most questions the majority of candidates understood which method to use. Some candidates omitted to draw a suitable diagram or to annotate the given diagram, and this resulted in writing incorrect equations. This was particularly the case in **Question 2**, **Question 6** and **Question 7**.

Candidates are reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra. In questions where candidates are asked to show that a result is true, they should not quote formulae, but rather derive them from first principles, as in **Question 6(a)**.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments equations and conservation of energy equations. Errors in dimensions were often seen in **Question 2** and **Question 3**.

Comments on specific questions

Question 1

The majority of candidates were successful on this question. Some candidates made sign errors, usually when they had not drawn a diagram. Other candidates made the incorrect assumption that the final velocity is equal to zero rather than that the tension is equal to zero when the string first becomes slack.

Question 2

Most candidates approached this question by taking moments about *AB* and setting up a three-term equation. Frequently, errors occurred in taking moments, either with masses or distances, and the resulting equations were often dimensionally incorrect. There were also a high number of algebraic errors in the simplification of the moments equation.

The majority of candidates realised that for equilibrium $\bar{x} > x$, but most did not include also the equality situation. The second solution of the quadratic equality obtained, $\bar{x} > 6a$, was often not excluded in the candidate's final answer.

A number of candidates approached this question by finding the centre of mass of the triangle *ADC* with *B* as the origin. This was a very neat and simple method for solving the problem.

Question 3

Almost all candidates knew that they needed to form and solve a conservation of energy equation, involving kinetic energy, elastic potential energy and gravitational potential energy. There were a few common errors in solutions. A significant minority of candidates did not consider the initial equilibrium position and so did not find the extension of the string when P begins it motion. This led to one instead of two elastic potential energy terms in the energy equation. Of those candidates who did realise that they needed the elastic potential energy when the extension is a and when it is $\frac{3}{4}a$, some went on to combine the two terms incorrectly by saying $a^2 - \left(\frac{3a}{4}\right)^2 = a^2\left(1-\frac{3}{4}\right) = \frac{a^2}{4}$.

A variety of dimensional errors in the energy equation were seen. Candidates are advised to check carefully that all the terms in their equation have the same dimensions.

Question 4

(a) The majority of candidates set up a differential equation from Newton's second law of motion and then separated the variables. Most candidates then integrated into a logarithmic term, often correctly, but sometimes with numerical errors. The constant of integration was found and then the solution was rearranged so that v^3 was the subject. Some candidates left this as their final answer, without taking the cube root to find v in terms of x.

(b) The majority of candidates found the maximum value of v from their answer to part **(a)**. It is worth noting that the answer can also be found by equating the acceleration in part **(a)** to zero and solving for v. This latter method is preferable because it does not depend on the solution of the differential equation being correct.

Question 5

(a) This part was answered well. A few candidates omitted intermediate lines of working and were not awarded full marks. It is important to remember that when an answer is given, each step in the working must be shown. In this case, moving from $\frac{1}{\cos^2\theta}$ to $\left(1+\tan^2\theta\right)$ requires the intermediate step $\sec^2 x$. A small minority of candidates quoted the given result from the formula book without any working.

(b) This part was answered very well by almost all candidates. Candidates had no problem in substituting the given values into the trajectory equation to find u and then solving the resulting quadratic equation.

Question 6

About half of the candidates fared well on this question. The remaining half of the candidates found the question challenging and did not make much progress in part **(a)**. A significant number, who did not derive the given result, did not attempt part **(b)**.

(a) A significant number of candidates made the incorrect assumption that the radius of the circular motion would be the same for the two possible values of angle *ARB*. Some candidates did not make it clear what the symbols in their equations represented. They needed to make it very clear in their working or on the diagram what they intended the radius to be. Many candidates found equations involving *N* which were not needed in this part.

Some candidates who wrote down incorrect equations managed to arrive at the given result with sines instead of cosines.

(b) Candidates were more successful on this part compared with the previous part and there were a good number of correct answers from those who continued even though they could not derive the result in part **(a)**. It is worth noting that a result is often given in one part of a question so that candidates can continue to the next part of the question using that result. The most common error in this part was to find only one of *N* and $\cos\alpha$. Finding the other quantity would have involved only a simple substitution.

Question 7

(a) Almost all candidates knew that they needed to write down two equations, one for conservation of momentum and one for Newton's law of restitution. The common errors were including incorrect masses, having inconsistent signs for the directions of the velocities and algebraic slips in solving to find the speeds. Almost all candidates understood that $\alpha + \beta = 90°$, so $\cos\beta = \sin\alpha$.

(b) Only about 40 per cent of candidates completed this part successfully, with 20 per cent not making an attempt. Others who followed the correct method with their incorrect result from part **(a)** gained partial credit. A common error was to have the reciprocal of the correct expression for $\tan\alpha$.

FURTHER MATHEMATICS

Paper 9231/33
Paper 3

Key messages

A diagram is often an invaluable tool in helping candidates to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates may draw their own diagram as well.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that there is no doubt that the offered solution is clear and complete. However, in all questions candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

Candidates should be encouraged to draw a suitable diagram or, in case a diagram is provided, to annotated it, as this helps understand the problem and model it correctly. For example, in **Question 3**, candidates who correctly annotated the diagram provided, realised that there were five forces involved and were able to determine the best point about which to work out the moments equation in **(a)**, as well as to correctly write the equations of forces in **(b)**.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's second law, e.g. to set up a differential equation or in questions involving collisions, they must ensure they explicitly mention the mass or masses involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra, and not to quote formulae, but rather to derive them from first principles, as in **Question 6a**.

Comments on specific questions

Question 1

Most candidates were able to work out the correct expression for the angular velocity. Stronger candidates then used the angular velocity to obtain the correct answer, without determining the period of revolution. A common mistake was to omit the multiplication by 60, therefore obtaining the number of revolutions per second.

Question 2

(a) Many candidates correctly identified expressions for the kinetic and elastic potential energy in the two situations, equated them and obtained the correct answers, often in a very elegant way. Some candidates attempted a variety of incorrect approaches (SUVAT, equilibrium of forces), which invariably led to an incorrect answer and rarely gained any credit. A common mistake was to omit the coefficient 2 in the denominator of the elastic potential energy, or to incorrectly simplify one of the terms of the elastic potential energy into mga or $\frac{1}{4}mga$ instead of $\frac{1}{2}mga$ or, respectively, $\frac{1}{8}mga$.

(b) This question proved more challenging than the previous one. Common mistakes were in the expression for the extension of the string, in the use of Hooke's formula (e.g., including a factor of 2 in the denominator, or adding an additional term $-mg$) or, as for **(a)** in the use of an incorrect approach, often SUVAT. More efficient approaches reached the solution in two well-presented steps.

Question 3

(a) Candidates who wrote the equation of the moments about point *A* often obtained the correct answer. Those who chose another point very rarely managed to gain credit, as their equations usually did not contain the correct number of terms. Stronger candidates annotated the diagram provided with an indication of the forces acting at the different points and stated explicitly the point about which they were taking the moments. Candidates are strongly encouraged to annotate the diagram, especially in situations where there are many forces acting at different points, as well as to explicitly state the point about which they are taking moments.

(b) This question proved challenging for most candidates who attempted to resolve forces in only one direction, vertical, horizontal, parallel, or perpendicular to the rod. Often, they did not include all components, and this would have been evident had they annotated the diagram. Stronger candidates realised that the best strategy was to resolve forces horizontally and vertically. They usually wrote the correct system of equations and obtained the correct answer, often showing strong algebraic manipulative skills.

Question 4

(a) Most candidates wrote the correct differential equation and separated the variables. However, only some of them used partial fractions to integrate the equations and only stronger candidates obtained the correct solution. These candidates almost always determined the correct value for the constant of integration from the boundary condition and then obtained the correct answer, showing mastery in the manipulation of logarithms.

(b) Even though the correct answer could be obtained without integrating the equation in **(a)**, very few candidates who did not do so were able to score the mark allocated to this part question. However, candidates who did well in **(a)**, usually had no problems in reaching the correct answer.

Question 5

(a) To obtain the correct answer for this question, candidates were required to perform three steps.

1. Use the equation for the conservation of energy at the endpoints.
2. Use Newton's second law at point.
3. Solve the resulting system of simultaneous equations to obtain the values of k and θ.

Most candidates realised this and attempted to write the equations, with various degrees of success. A common mistake was to invert the sign for the gravitational potential energy in the energy equation and/or the term due to gravity in Newton's second law, thus writing $\frac{1}{2}mkag + mga\cos\theta = \frac{1}{6}mag$ and/or, respectively, $\frac{11}{6}mg + mg\cos\theta = mkg$.

(b) Candidates used a variety of approaches to answer this question. They mostly involved one or more SUVAT equations, with a few candidates choosing the conservation of energy. Even though the majority of approaches were correct, only stronger candidates selected the SUVAT equation $v^2 = u^2 + 2as$, which quickly led to the correct answer. Some candidates opted for the more complicated solution of a system of two SUVAT equations involving time, e.g. $v = u + at$ and $s = ut + \frac{1}{2}at^2$, often obtaining an incorrect answer.

Question 6

(a) Many candidates found this question rather challenging. Most candidates wrote the momentum equation correctly, but only a few could then write the corresponding equation for Newton's elastic law with consistent signs for the velocities involved. Stronger candidates were able to write two correct equations and solved them flawlessly, often in an elegant manner.

(b) Most candidates understood that this question required them to calculate the initial and final energy and subtract the former from the latter to score the method mark. Many realised that the component of the velocity of particle B perpendicular to the lines of centre was $\frac{5}{8}u\ \sin(\alpha)$. The majority of stronger candidates who scored full marks in **(a)** scored full marks in this question too.

Question 7

(a) In this question candidates were asked to show that the total time of flight was equal to a given expression. Most of them realised that as the answer was given, they were expected to show the result from first principles and to carefully explain each step.

(b) This question also asked candidates to explain a given result. The key concept to express was that the vertical component of the velocity is not affected by the impact, as the wall is vertical. Many candidates were able to explain this, and some gave more elaborated explanations that showed a deep understanding of the underlying concepts.

(c) To obtain the correct answer for this question, candidates were required to perform three steps:

1. Calculate the time taken to hit the wall using the horizontal component of the velocity.
2. Calculate the time taken to hit the ground after hitting the wall, again using the horizontal component of the velocity, after the collision.
3. Add the two times, equate the sum to the expression provided in **(a)**, and solve the equation.

Many candidates were able to determine the horizontal component of the speed before and after the collision, but only the strongest used this information, along with the fact that the barrier was 15 metres from point O, to obtain the expressions for the times. Most of those who did, and who wrote the correct equation, had no difficulty solving the equation. Stronger candidates did this without evaluating $\cos(75°)$ and $\sin(75°)$ explicitly, but instead used the double angle formula.

FURTHER MATHEMATICS

Paper 9231/41
Paper 4 Further Probability & Statistics

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that the offered solution is clear and complete.

Particular care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' and not 'the test proves that….'.

General comments

The standard was of responses was generally good. Many candidates presented clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case where several different values are calculated, each depending on the previous one. Candidates are encouraged to show the detail of intermediate stages in their working so that partial credit can be awarded if appropriate. This was particularly the case in **Question 6**.

Comments on specific questions

Question 1

The majority of candidates answered this question correctly. The most common error was to use a t-value instead of the z-value required when the sample sizes are large. A minority of candidates used a pooled variance, without stating the assumption they were making in assuming this. For this approach to be valid, it must be assumed that the population variance for the heights of the trees in forests in two different regions are equal. There are no grounds for believing this to be true.

Question 2

(a) This part posed few problems, although the instruction to give answers correct to 3 decimal places was not always followed.

(b) Almost all candidates recognised the need for a chi-squared test, but few were completely successful. Since the value of p was found to be 0.098 in part **(a)**, the expected (and observed) frequencies in the first three columns needed to be combined, as well as those in the last two columns, before calculating the contributions to the test statistic. Many candidates either combined some of the relevant columns or none at all. The hypotheses were often stated with insufficient detail, for example 'it is a good fit model' or 'the distribution is a good fit'. It is expected that both the distribution and the data are mentioned in the hypotheses. Examples are 'the binomial distribution B(8, 0.6) is a good model for the data' or, as a minimum, 'the distribution B(8, 0.6) fits the data'. A minority of candidates did not include any level of uncertainty in their conclusion.

A minority of candidates offered minimal working for their calculation of the test statistic, lacking any values for the expected frequencies or the contributions to the test statistic. The rubric for this paper states that all necessary working must be shown, so a single value stated for the test statistic is insufficient to gain full marks.

Question 3

(a) This question required an application of the Wilcoxon matched-pairs signed-rank test. Most candidates found the signed differences and the signed ranks for the given data, and successfully evaluated the test statistic as 33. This was then compared with the critical value of 25 and a conclusion drawn.

A significant minority of candidates did not write down the correct hypotheses. A common error was to use 'mean' instead of 'median' either in words or by using the symbol μ. When using words, it is essential to use the phrase 'population median' rather than just 'median'. Some candidates were not clear whether the test was a one-tail test or a two-tail test. Some candidates misinterpreted the conclusion of the test while others did not express any uncertainty in the language used in their conclusion.

(b) A wide variety of reasons were given by candidates in this part, but few of them were correct. That the population distribution of the differences is unknown is the crucial fact which renders use of a paired *t*-test as inappropriate.

Question 4

The majority of candidates answered this question well.

(a) This part was almost always answered correctly.

(b) This part was answered well by most candidates.

(c) Almost all candidates differentiated their probability generating function for Z and used the fact that $E(Z)$ is the value of this differential when $t = 1$.

Question 5

(a) Most candidates equated the given cumulative distribution function to 0.75 and solved to find the correct value of X.

(b) Solving this part required use of the formula $\text{Var}\left(X^2\right) = E\left(X^4\right) - E^2\left(X^2\right)$. The majority of candidates first evaluated the two expressions on the right-hand side, each of which involved integration, and obtained the correct answer.

(c) Almost all candidates successfully changed the variable in the cumulative distribution function and then differentiated to find the probability distribution function. Some candidates were confused as to whether to differentiate or integrate and others did not change the domain to match the change in variable.

Question 6

Most candidates performed well on this question, with a correct calculation of the test statistic as 0.640, a correct comparison with the critical *t*-value 2.086 and a correct conclusion to accept H_0. Sometimes, the final conclusion was not interpreted in context or there was no uncertainty in the language used in the conclusion. A minority of candidates did not show sufficient working in their calculation of the test statistic and the available partial credit could not be awarded when the calculated value was incorrect. The rubric for this paper states that sufficient working must be shown, so candidates are advised that it is not a good strategy to omit working and to only give an answer from their calculator.

The question states that the diameters of the pipes produced by the two machines are assumed to be normally distributed with equal population variances. This is a clear indication that a pooled variance needs to be used. A minority of candidates did not make use of this assumption and calculated and used a two-sample variance applicable to distributions with unequal population variances.

FURTHER MATHEMATICS

Paper 9231/42
Paper 4 Further Probability & Statistics

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that the offered solution is clear and complete.

Particular care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' and not 'the test proves that….'.

General comments

Almost all candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case where several different values are calculated, each depending on the previous one. Candidates are encouraged to show the detail of intermediate stages in their working so that partial credit can be given if appropriate. This was particularly the case in **Question 3**.

Comments on specific questions

Question 1

Responses to this question varied from concise and accurate solutions to attempts that covered two pages and involved increasing amounts of algebra with no final solution. There are essentially three steps to solving the problem. The first is to use the definition of a confidence interval to form two equations which can be added to give the value of the sample mean, and hence $\sum x$. The second step is to subtract the two equations to give the value of the sample variance. The final step is to find $\sum x^2$ from the expression.

Those who employed this approach were usually successful in solving the problem, though it was a common error for either a z-value (usually 1.645) or an incorrect t-value (usually 1.383) to be used.

Some candidates opted to combine all the relevant formulae before substituting any numerical values. This approach was rarely successful because of algebraic errors along the way. Other candidates chose to set up simultaneous equations in the two quantities to be found: $\sum x$ and $\sum x^2$. This approach usually led to pages of working without final answers.

Question 2

Many candidates scored full marks for this question, with a fully correct solution including a correct statement of the hypotheses and a conclusion with an appropriate degree of uncertainty in the language used. A minority of candidates gave only the null hypothesis without the alternative hypothesis. Other candidates gave their final conclusion as a definite statement, for example, 'the performance is independent of region'. It is worth noting that a test result does not 'prove' anything, it merely indicates whether there is 'sufficient evidence to suggest' or the equivalent.

A minority of candidates offered minimal working for their calculation of the test statistic, lacking any values for the expected frequencies or the contributions to the test statistic. The rubric for this paper states that all necessary working must be shown, so a single value stated for the test statistic is insufficient.

Question 3

Most candidates performed well on this question, with a correct calculation of the test statistic as 1.75, a correct comparison with the critical z-value 1.645 and a correct conclusion to reject H_0. The final mark often could not be awarded either because the final conclusion was not interpreted in context or because there was no uncertainty in the language used in the conclusion. A minority of candidates did not show sufficient working in their calculation of the test statistic and the available partial credit could not be awarded when the calculated value was incorrect.

The question did not give any reason to suggest that a pooled variance was appropriate, but a minority of candidates used it, without stating the assumption they were making, namely that the two sample variances were equal.

Question 4

(a) This part was almost always answered correctly, with sufficient working shown to justify the given answer.

(b) This part was less well answered. Most candidates were able to find the lower quartile for the function, but there was confusion about how to find the upper quartile. The main problem seemed to be in knowing how to deal with the piecewise nature of the probability density function. The most successful approach was to find the cumulative distribution function F and use it find the value of x for which F is equal to 0.75.

(c) This part was answered well by most candidates. A minority of candidates made an error in the integration, others in the use of the formula for variance.

Question 5

The vast majority of candidates answered this question well.

(a) This part was almost always answered correctly.

(b) This part was answered well by most candidates. A minority of candidates assumed that the second probability generating function was the same as the first.

(c) Almost all candidates differentiated their probability generating function for Z twice and used the correct formula for the variance.

(d) The most probable value of Z is the power of t in the probability generating function which has the greatest coefficient. Most candidates knew this and earned the mark following through on their probability generating function. A minority of candidates identified the correct term but thought that the answer was the value of the greatest coefficient.

Question 6

(a) The majority of candidates knew how to apply the Wilcoxon rank-sum test. They ranked all the data from companies *A* and *B* in order from 1 to 24 and then summed the ranks for the two companies individually. Most candidates did this accurately and identified the test statistic as 127 from the two sums 127 and 173. Those candidates who drew up a table with all the relevant values were usually more successful in working accurately than those who wrote around the table of data. Either approach is fine, but in this test, accuracy is paramount.

Because of the sizes of the samples involved, a normal approximation is required. Most candidates realised this and found the mean and variance of the relevant distribution from the usual formulae. The next step is to standardise the test statistic. A continuity correction is required and it was a common error for this to be omitted. The final step is a comparison of *z*-values or of probabilities. At this stage there was confusion about whether this was a one-tail or two-tail test, although the question states quite clearly that the manager's claim is that one set of employees earn more than the other set.

A significant minority of candidates did not write down the correct hypotheses. A common error was to use 'mean' instead of 'median' either in words or by using the symbol μ. When using words, it is essential to use the phrase 'population median' rather than just 'median'.

A minority of candidates used a Wilcoxon matched-pairs signed-rank test instead of a Wilcoxon rank-sum test, treating the given data as matched-pairs. Since the data concerns employees from different companies, the pairs are not matched.

(b) The given data represents earnings from two different companies, so the individuals in the samples cannot be paired up. This means that any paired sample test is not valid. A common statement that was seen was 'a paired *t*-test cannot be used because it is not a paired sample'. Some reference to why it was not a paired sample was expected.

FURTHER MATHEMATICS

Paper 9231/43
Paper 4 Further Probability & Statistics

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that the offered solution is clear and complete.

Particular care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' and not 'the test proves that….'.

General comments

The standard was of responses was generally good. Many candidates presented clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case where several different values are calculated, each depending on the previous one. Candidates are encouraged to show the detail of intermediate stages in their working so that partial credit can be awarded if appropriate. This was particularly the case in **Question 6**.

Comments on specific questions

Question 1

The majority of candidates answered this question correctly. The most common error was to use a t-value instead of the z-value required when the sample sizes are large. A minority of candidates used a pooled variance, without stating the assumption they were making in assuming this. For this approach to be valid, it must be assumed that the population variance for the heights of the trees in forests in two different regions are equal. There are no grounds for believing this to be true.

Question 2

(a) This part posed few problems, although the instruction to give answers correct to 3 decimal places was not always followed.

(b) Almost all candidates recognised the need for a chi-squared test, but few were completely successful. Since the value of p was found to be 0.098 in part **(a)**, the expected (and observed) frequencies in the first three columns needed to be combined, as well as those in the last two columns, before calculating the contributions to the test statistic. Many candidates either combined some of the relevant columns or none at all. The hypotheses were often stated with insufficient detail, for example 'it is a good fit model' or 'the distribution is a good fit'. It is expected that both the distribution and the data are mentioned in the hypotheses. Examples are 'the binomial distribution B(8, 0.6) is a good model for the data' or, as a minimum, 'the distribution B(8, 0.6) fits the data'. A minority of candidates did not include any level of uncertainty in their conclusion.

A minority of candidates offered minimal working for their calculation of the test statistic, lacking any values for the expected frequencies or the contributions to the test statistic. The rubric for this paper states that all necessary working must be shown, so a single value stated for the test statistic is insufficient to gain full marks.

Question 3

(a) This question required an application of the Wilcoxon matched-pairs signed-rank test. Most candidates found the signed differences and the signed ranks for the given data, and successfully evaluated the test statistic as 33. This was then compared with the critical value of 25 and a conclusion drawn.

A significant minority of candidates did not write down the correct hypotheses. A common error was to use ‘mean’ instead of ‘median’ either in words or by using the symbol μ. When using words, it is essential to use the phrase ‘population median’ rather than just ‘median’. Some candidates were not clear whether the test was a one-tail test or a two-tail test. Some candidates misinterpreted the conclusion of the test while others did not express any uncertainty in the language used in their conclusion.

(b) A wide variety of reasons were given by candidates in this part, but few of them were correct. That the population distribution of the differences is unknown is the crucial fact which renders use of a paired *t*-test as inappropriate.

Question 4

The majority of candidates answered this question well.

(a) This part was almost always answered correctly.

(b) This part was answered well by most candidates.

(c) Almost all candidates differentiated their probability generating function for *Z* and used the fact that E(*Z*) is the value of this differential when $t = 1$.

Question 5

(a) Most candidates equated the given cumulative distribution function to 0.75 and solved to find the correct value of *X.*

(b) Solving this part required use of the formula $\mathrm{Var}\left(X^2\right) = E\left(X^4\right) - E^2\left(X^2\right)$. The majority of candidates first evaluated the two expressions on the right-hand side, each of which involved integration, and obtained the correct answer.

(c) Almost all candidates successfully changed the variable in the cumulative distribution function and then differentiated to find the probability distribution function. Some candidates were confused as to whether to differentiate or integrate and others did not change the domain to match the change in variable.

Question 6

Most candidates performed well on this question, with a correct calculation of the test statistic as 0.640, a correct comparison with the critical *t*-value 2.086 and a correct conclusion to accept H_0. Sometimes, the final conclusion was not interpreted in context or there was no uncertainty in the language used in the conclusion. A minority of candidates did not show sufficient working in their calculation of the test statistic and the available partial credit could not be awarded when the calculated value was incorrect. The rubric for this paper states that sufficient working must be shown, so candidates are advised that it is not a good strategy to omit working and to only give an answer from their calculator.

The question states that the diameters of the pipes produced by the two machines are assumed to be normally distributed with equal population variances. This is a clear indication that a pooled variance needs to be used. A minority of candidates did not make use of this assumption and calculated and used a two-sample variance applicable to distributions with unequal population variances.