FURTHER MATHEMATICS

Paper 9231/11
Further Pure Mathematics 11

Key messages

- Candidates should read every question very carefully so that they answer all aspects in adequate depth.
- Good solutions show all the steps, particularly when proving a given result.
- All sketch graphs need to be fully labelled and carefully drawn to show significant points and behaviour at limits.
- When working with inequalities candidates need to be sure they are not multiplying by a quantity which could be negative.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations. It seemed that almost all were able to complete the paper in the time allowed.

Comments on specific questions

Question 1

(a) Good solutions derived the result $(r+1)^2 - r^2 = 2r+1$ and then showed clearly the summation of both sides to work to the known result. Weaker candidates summed the left-hand side using the method of differences but struggled to link this to $\sum_{r=1}^{n} r$.

(b) This was often well done, with the majority of errors being in the algebra or not simplifying the final result.

Question 2

The base case was usually demonstrated to be true and the statement made. The hypothesis needed to be a statement connecting the sum of terms with the algebraic form given. Some weaker candidates tried to prove that the kth term of the series was equal to $\frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$. The majority added the (k + 1)th term to the sum to k terms as given and produced convincing algebra to show that the result holds for (k +1). The most complete solutions concluded by saying that because the result is true for n = 1, and if it is true for n = k then it is also true for n = k +1, then it holds for all positive integers values of n.

Question 3

(a) The majority of candidates were successful with this question. They understood the relationship between roots and coefficients, and errors were usually arithmetical.

(b) The most successful method was to write down the equation with each root substituted and sum them.

Question 4

In all vector questions candidates should check their working carefully as an error in arithmetic can affect the structure of the question. They also need to be clear on the difference between a position vector and a direction vector.

(a) Finding the direction of the common perpendicular was well done. The majority of candidates then found a vector joining a point on each line and projected it on to the normal to accurately find the distance. A small number tried to find where the common perpendicular intersects each line, but these attempts were rarely complete.

(b) A few candidates confused the position vector of the given point with a direction vector. Successful solutions found a second direction using a point on the line l_1.

Question 5

(a) Those who found the value of k first had slightly easier arithmetic. Matrix multiplication was usually done well. In questions of this type candidates should show the result of multiplying together one pair of matrices before going on to multiply by the third.

(b) This question was answered well with very few looking for invariant points or considering lines that do not pass through the origin.

(c) Candidates clearly know the form of the matrices representing enlargement and one-way stretch. A few found the matrix for reflection more difficult. Problems arose when candidates used the same unknown in both **D** and **E** and so were unable to solve their equations. Usually, the question was well answered.

Question 6

(a) Most proofs were complete. The best solutions stated that $r \neq 0$ before dividing.

(b) The majority of candidates found the result $\theta = \frac{\pi}{6}$. For a complete solution the possibility that $\cos\theta = 0$ should have been considered. Candidates are reminded to express polar coordinates in the form (r, θ)

(c) Most diagrams were of acceptable shape and labelled correctly. Good sketches clearly showed C_1 as a semi-circle with P on the right-hand side of C_1.

(d) Joining the line *OP* on the diagram from **part (c)** showed candidates with good sketches that the required area needed to be found in two parts. Strong candidates choose the correct limits and added the two parts. Most candidates could integrate the functions correctly, and maintained accuracy when applying the required identities for $\cos^2\theta$ and $\sin^2 2\theta$.

Question 7

(a) The vertical asymptotes were almost always correct.

(b) The stationary points were usually correct, with only a few errors in arithmetic.

(c) There were some very good clear sketches, with asymptotes and axes labelled. A correctly shaped curve needed to show that the left-hand branch crossed the horizontal asymptote, had a clear minimum and then approached the asymptote from below.

(d) The strongest candidates could sketch $y = \dfrac{1}{f(x)}$ by reference to the graph of $y = f(x)$. Good sketches used $y = 1$ as the only asymptote. They also made clear the intersection at (–1, 0), the maximum point and that the graph then approached the asymptote from above.

(e) Those candidates who tried to work with the inequality had little success unless they considered both cases for the sign of $(x^2 - x - 2)$. When multiplying inequalities by variable terms, candidates should be more aware of the need to establish the sign of the multiplication term. The best solutions used equations to find the critical values, although the value $x = 0$ was sometimes omitted. The strongest candidates then used their sketches, rather than trying to manipulate inequalities algebraically, to determine the correct regions between the critical values. This also allowed them to exclude the asymptotes, as required.

FURTHER MATHEMATICS

Paper 9231/12
Further Pure Mathematics 12

Key messages

- Candidates should read every question very carefully so that they answer all aspects in adequate depth.
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- All sketch graphs need to be fully labelled and carefully drawn to show significant points and behaviour at limits.
- When working with inequalities candidates need to be sure they are not multiplying by a quantity which could be negative.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations. It seemed that almost all were able to complete the paper in the time allowed.

Comments on specific questions

Question 1

(a) This was usually well done, with only a few making errors in expanding the brackets and collecting like terms.

(b) The expansion of $(r-1)^3$ was required for candidates to properly show the relationship given in the question but it was frequently omitted. Some tried to use partial fractions but were usually unsuccessful as they chose an incorrect structure for the numerators. The method of differences was used correctly and almost all candidates showed enough terms to justify the result.

(c) This was answered well.

Question 2

Most candidates understand the structure of a proof by induction but need to be sure that their statements are accurate. For the base case, a full solution needed the result of differentiating once to be shown to be the same as substituting $n = 1$ into the given formula. The inductive step was a straightforward application of the chain rule and the best solutions showed, and stated, that this was the required result for $k + 1$. The most complete solutions concluded by saying that because the result is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$, then it holds for all positive integers values of n.

Question 3

(a) Almost all candidates understood the order in which the transformations were applied. To gain full marks they needed to use the terms ‘shear’ and one-way ‘stretch’.

(b) A surprising number of candidates did not attempt to find the inverse of matrix M. Those who did were usually correct. Only the strongest recognised that the area of *OPQR* must be positive and gave the area as $|k|$.

(c) The majority decided to show that the invariant line through the origin has the gradient $\frac{1}{k-1}$ and managed to do this efficiently, with hardly any considering invariant points. Those who started with a point on the line needed to show clearly that its image lies on the same line, and sometimes became tangled in the algebra.

Question 4

A few candidates became confused between equations but the methods are, in general, well-understood.

(a) Most used the substitution method and showed the expansion of $(y-1)^3$ fully to justify their arrival at the given answer. Some found the coefficients of the new equation using the sums and products from the original equation and this also worked effectively in this case.

(b) As long as they used the right equation the sum of the squares of the roots was found accurately using the standard method.

To find S_3 most used the relationship $S_3 - S_2 + S_1 - 2S_0 = 0$ and errors were usually caused by saying $S_0 = 1$.

A few tried to use formulae but often remembered them incorrectly.

(c) The value of S_{-1} was mostly correct, either from the coefficients of the equation or from the relationship $S_2 - S_1 + S_0 - 2S_{-1} = 0$. Those who used a similar method for S_{-2} were usually successful. Those who attempted to remember a formula often missed a term or used a wrong sign.

Question 5

As always in vector questions, an error in arithmetic at the start can make a great change to the demands of the question.

(a) This was usually correct, with the vector product method being most popular and a few using simultaneous equations.

(b) A variety of methods were used and the best solutions explained which they were using, the significance of the numerical work and checked that the line does not lie in the plane. Those who tried to find the intersection of the line and plane did not always make it clear or understand that they had a contradiction which showed that there was no intersection.

A common error was to attempt to show that the direction of the line and the normal to the plane are parallel – instead of showing that they are perpendicular.

(c) Again, there were many methods used, and it was not always made clear which one was being attempted. Those who found a vector joining a point on the line to one on the plane and took the scalar product with the unit normal were usually correct. Others used the formula by substituting one point on the line into the equation of the plane. Here a very common error was to forget that the plane had to be arranged to the form $x - 4y + 3z + 1 = 0$ for this to be valid.

(d) The basic method is well-understood. Candidates who found a point common to both planes by choosing one coordinate as zero were often correct, although it is always worth checking that the point does in fact lie on both planes. The direction was found either by using the vector product or by finding the vector joining two common points. A number of candidates lost the final mark by not expressing the answer as an equation.

Question 6

(a) Most candidates drew the basic shape for the graph but only the strongest made sure that the behaviour at the pole was correct. The greatest distance was often stated correctly.

(b) The question asked for the exact area and to gain full marks all stages in the working should be seen. After the brackets were expanded there were three terms to integrate, and a surprising error was to ignore the $e^{-\pi}$ term.

(c) A majority of candidates knew the expression to differentiate and used the product rule correctly. Most remembered to attempt the last part and the numerical values of the substitutions were shown to demonstrate the sign change.

Question 7

(a) This familiar question was well answered, with most candidates continuing the division process to find the constant term in the equation of the oblique asymptote.

(b) The stationary points were usually correct.

(c) There were some very good clear sketches, with asymptotes and axes labelled and the curve of the correct shape.

(d) The strongest candidates could write down the stationary point of $y = \frac{1}{f(x)}$ by reference to the graph of $y = f(x)$. Those who differentiated usually found the point $\left(-2, -\frac{1}{4}\right)$ and most realised that the point (0, 0) is not a point of the graph and ignored it.

(e) Good sketches used both axes as asymptotes. They also made clear the intersection at (–1, 0), the minimum point found in **part (d)** and that the graph then approached the negative *x* axis from below.

The best solutions used an equation to find the critical values. Many tried to use inequalities with little success unless they considered both cases for the sign of $(x+1)$. When multiplying inequalities by variable terms, candidates should be more aware of the need to establish the sign of the multiplication term.

Only the strongest candidates realised that for $x < -1$ the lower branch of $y = f(x)$ lies below the graph of $y = \frac{1}{f(x)}$ and so included it in the final inequality.

FURTHER MATHEMATICS

Paper 9231/13
Further Pure Mathematics 13

Key messages

- Candidates should read every question very carefully so that they answer all aspects in adequate depth.
- Good solutions show all the steps, particularly when proving a given result.
- All sketch graphs need to be fully labelled and carefully drawn to show significant points and behaviour at limits.
- When working with inequalities candidates need to be sure they are not multiplying by a quantity which could be negative.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations. It seemed that almost all were able to complete the paper in the time allowed.

Comments on specific questions

Question 1

(a) Good solutions derived the result $(r+1)^2 - r^2 = 2r + 1$ and then showed clearly the summation of both sides to work to the known result. Weaker candidates summed the left-hand side using the method of differences but struggled to link this to $\sum_{r=1}^{n} r$.

(b) This was often well done, with the majority of errors being in the algebra or not simplifying the final result.

Question 2

The base case was usually demonstrated to be true and the statement made. The hypothesis needed to be a statement connecting the sum of terms with the algebraic form given. Some weaker candidates tried to prove that the kth <u>term</u> of the series was equal to $\dfrac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$. The majority added the (k + 1)th term to the sum to k terms as given and produced convincing algebra to show that the result holds for (k +1). The most complete solutions concluded by saying that because the result is true for n = 1, and if it is true for n = k then it is also true for n = k +1, then it holds for all positive integers values of n.

Question 3

(a) The majority of candidates were successful with this question. They understood the relationship between roots and coefficients, and errors were usually arithmetical.

(b) The most successful method was to write down the equation with each root substituted and sum them.

Question 4

In all vector questions candidates should check their working carefully as an error in arithmetic can affect the structure of the question. They also need to be clear on the difference between a position vector and a direction vector.

(a) Finding the direction of the common perpendicular was well done. The majority of candidates then found a vector joining a point on each line and projected it on to the normal to accurately find the distance. A small number tried to find where the common perpendicular intersects each line, but these attempts were rarely complete.

(b) A few candidates confused the position vector of the given point with a direction vector. Successful solutions found a second direction using a point on the line l_1.

Question 5

(a) Those who found the value of k first had slightly easier arithmetic. Matrix multiplication was usually done well. In questions of this type candidates should show the result of multiplying together one pair of matrices before going on to multiply by the third.

(b) This question was answered well with very few looking for invariant points or considering lines that do not pass through the origin.

(c) Candidates clearly know the form of the matrices representing enlargement and one-way stretch. A few found the matrix for reflection more difficult. Problems arose when candidates used the same unknown in both **D** and **E** and so were unable to solve their equations. Usually, the question was well answered.

Question 6

(a) Most proofs were complete. The best solutions stated that $r \neq 0$ before dividing.

(b) The majority of candidates found the result $\theta = \frac{\pi}{6}$. For a complete solution the possibility that $\cos\theta = 0$ should have been considered. Candidates are reminded to express polar coordinates in the form (r, θ)

(c) Most diagrams were of acceptable shape and labelled correctly. Good sketches clearly showed C_1 as a semi-circle with P on the right-hand side of C_1.

(d) Joining the line OP on the diagram from **part (c)** showed candidates with good sketches that the required area needed to be found in two parts. Strong candidates choose the correct limits and added the two parts. Most candidates could integrate the functions correctly, and maintained accuracy when applying the required identities for $\cos^2\theta$ and $\sin^2 2\theta$.

Question 7

(a) The vertical asymptotes were almost always correct.

(b) The stationary points were usually correct, with only a few errors in arithmetic.

(c) There were some very good clear sketches, with asymptotes and axes labelled. A correctly shaped curve needed to show that the left-hand branch crossed the horizontal asymptote, had a clear minimum and then approached the asymptote from below.

(d) The strongest candidates could sketch $y = \dfrac{1}{f(x)}$ by reference to the graph of $y = f(x)$. Good sketches used $y = 1$ as the only asymptote. They also made clear the intersection at (–1, 0), the maximum point and that the graph then approached the asymptote from above.

(e) Those candidates who tried to work with the inequality had little success unless they considered both cases for the sign of $(x^2 - x - 2)$. When multiplying inequalities by variable terms, candidates should be more aware of the need to establish the sign of the multiplication term. The best solutions used equations to find the critical values, although the value $x = 0$ was sometimes omitted. The strongest candidates then used their sketches, rather than trying to manipulate inequalities algebraically, to determine the correct regions between the critical values. This also allowed them to exclude the asymptotes, as required.

FURTHER MATHEMATICS

Paper 9231/21
Further Pure Mathematics 21

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. It seemed that all were able to complete the paper in the time allowed. Sometimes candidates did not fully justify their answers and jumped to conclusions without justification, particularly where answers were given within the question. There were many scripts of a high standard.

Comments on specific questions

Question 1

Almost all candidates found the determinant correctly. While it was common to see row operations or elimination of a variable, only strong candidates gave a full geometric interpretation by emphasising that there are two non-identical parallel planes with a third plane intersecting both.

Question 2

The majority of candidates correctly stated the modulus and argument of $4+4\sqrt{3}\mathrm{i}$. Good candidates maintained accuracy when taking the cube root of the modulus and manipulating the argument, leaving the three roots in the required form.

Question 3

This was very well answered with most candidates accurately differentiating twice to find the Maclaurin's series, though a few accepted an erroneous value of f''(0) without checking their work.

Question 4

Almost all knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the particular integral and some errors when substituting initial conditions. A few candidates gave expressions instead of equations as their answer.

Question 5

(a) Most candidates accurately recalled the formula for the length of the curve with correct limits. Good candidates fully simplified $\sqrt{\dot{x}^2+\dot{y}^2}$ before substituting into the formula, which caused fewer errors and enabled a clear path to the answer.

(b) Almost all applied the chain rule to find the first derivative. Good candidates differentiated the first derivative with respect to t and applied the chain rule again to obtain $-\frac{t+1}{(t-1)^3}$. Very few realised that, given $t > 0$, this expression can only be positive when the denominator is negative.

Question 6

(a) Almost all worked from one side of the equation to the other, after writing cosh and sinh in terms of exponentials, to fully justify the given identity.

(b) Almost all used the given substitution with the identity from **part (a)** and integrated to get $\frac{4}{3}\sinh^3 x + \frac{4}{5}\sinh^5 x + C$, although it was common to see the arbitrary constant omitted.

(c) Most candidates used the correct method to find the integrating factor, and, after multiplying both sides of the equation by their integrating factor, were able to attempt to integrate the RHS. Good candidates maintained accuracy throughout, particularly when applying their result from **part (b)** and substituting in the initial conditions. Most candidates remembered to add an arbitrary constant after integrating the right-hand side, substituted the initial conditions and made y the subject.

Question 7

(a) This part of the question was well done, though a few candidates accepted zero eigenvectors without checking for errors in their working. A few candidates spent time finding $\det(\mathbf{A} - \lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix.

(b) Good candidates were able to maintain accuracy when substituting $\mathbf{A}$ into the characteristic equation, both when calculating $\mathbf{A}^2$ and when making $\mathbf{A}^{-1}$ the subject.

Question 8

(a) Most candidates stated the formula for the sum of the geometric progression accurately. Some wrote the exponent as $n-1$, which hindered their progress in **part (b)**. A few incorrectly stated that the sum was equal to zero.

(b) Almost all knew that de Moivre's theorem related the series to the sum of the geometric progression in **part (a)**. The strongest candidates took the imaginary part, after multiplying the numerator and denominator by the conjugate of $z-1$ or by $z^{-\frac{1}{2}}$. Use of appropriate trigonometric identities then led to the given answer.

(c) Most candidates formed a correct expression for the sum of the areas of the rectangles. Good candidates emphasised the substitution of $\theta = \frac{1}{n}$ into the equation from **part (b)**, clearly showing the given upper bound.

(d) Strong candidates correctly adapted their solution to **part (c)** and gave a suitable lower bound.

FURTHER MATHEMATICS

Paper 9231/22
Further Pure Mathematics 22

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed a thorough understanding of hyperbolic functions and de Moivre's theorem for complex numbers. It seemed that all were able to complete the paper in the time allowed. Sometimes candidates did not fully justify their answers and jumped to conclusions without justification, particularly where answers were given within the question. There were many scripts of a high standard.

Comments on specific questions

Question 1

Most candidates accurately differentiated twice using the chain and quotient rules. The few who instead applied the expansion of $\ln(1+x)$ given in the List of formulae (MF 19) were usually also successful. A common error was to not divide $f''(0)$ by 2 when writing out the Maclaurin's series.

Question 2

(a) The majority of candidates found the first derivative correctly using parametric differentiation.

(b) Most candidates were able to find $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ using the product rule, but some omitted to multiply by $-t^2$ to obtain the second derivative with respect to x as required.

Question 3

(a) After expanding $(\cos\theta + \mathbf{i}\sin\theta)^5$ using the binomial expansion, most candidates grouped together terms contributing to the real part before applying the identity $\cos^2\theta + \sin^2\theta = 1$ to fully justify the given result. Alternatively, a few candidates worked from the right-hand side to the left-hand side, applying $2\cos\theta = z + z^{-1}$, and were usually successful too.

(b) Almost all rearranged the polynomial equation correctly, linking with the previous part of the question by substituting $x = \cos\theta$. Solving $\cos 5\theta = \frac{1}{2}\sqrt{2}$, most candidates obtained $x = \cos\left(\frac{1}{20}\pi\right)$. A few candidates gave the values for θ rather than actual solutions, highlighting the need to read the question carefully. Although the four other solutions were often given in different ways, few candidates gave repeated solutions.

Question 4

Most candidates used the correct method to find the integrating factor, and, after multiplying both sides of the equation by their integrating factor, were able to attempt to integrate the right-hand side. Strong candidates maintained accuracy throughout, particularly when integrating by parts twice and substituting in the initial conditions. Most candidates remembered to add an arbitrary constant after integrating, substituted the initial conditions and made y the subject. A few candidates chose to use a different method, using a linear auxiliary equation and obtaining a complementary function of $y = Ae^{-3x}$. A particular integral was then found together with the appropriate values of the constants. A very small number of responses appeared to attempt to use the previous method, but introduced what to begin with was a constant term then later was treated as a variable term. No marks were given for this incorrect method.

Question 5

(a) Most candidates obtained a correct derivative in one form or another but failed to justify the given answer fully by stating $\operatorname{sech}^2 x \neq 0$, or equivalent. Almost all gained the independent mark for confirming the position of the root of the given equation.

(b) Almost all made a good attempt at the integration by parts with many obtaining $x \tanh x - \ln\cosh x$. Most converted to obtain a result in $\operatorname{sech} n$ and $\operatorname{sech} 1$. Some candidates did not make the comparison between the sum and the integral clear and there was the occasional error with inconsistent limits.

Question 6

Please note that due to a series-specific issue with question 6, full marks have been awarded to all candidates for this question to make sure that no candidates were disadvantaged.

Question 7

(a) The majority of candidates presented the proof of the identity clearly, working from one side to the other and labelling the variable consistently.

(b) Provided the correct initial formula was used, most candidates were able to gain a substantial amount of the marks. Some candidates had an inconsistent use of limits but were usually able to recover the correct limits in later work. Many were able to obtain an integrand of $\frac{\pi}{16}\sinh^2 u$, and make appropriate use of double angle formulae to convert to a correct integral. Having obtained $\frac{\pi}{32}\left[\frac{1}{4}\sinh 4u - u\right]_0^{\sinh^{-1}\frac{4}{3}}$, strong candidates showed enough working when substituting the limits to justify the given answer.

Question 8

(a) This was well done with candidates finding the first two derivatives of v or, alternatively, y with respect to x and substituting to justify the given equation. Good candidates clearly worked from the given $v-x$ equation to the given $y-x$ equation, keeping accurate when substituting and factorising.

(b) Almost all knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the constants and some problems with notation. A few candidates gave expressions instead of equations as their answer. Good candidates substituted $v = y^4$ and made y the subject.

FURTHER MATHEMATICS

Paper 9231/23
Further Pure Mathematics 23

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.
- Candidates should make use of results derived or given in earlier parts of a question.

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Question 4

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Question 5

(a) Most candidates accurately recalled the formula for the length of the curve with correct limits. Good candidates fully simplified $\sqrt{\dot{x}^2+\dot{y}^2}$ before substituting into the formula, which caused fewer errors and enabled a clear path to the answer.

(b) Almost all applied the chain rule to find the first derivative. Good candidates differentiated the first derivative with respect to t and applied the chain rule again to obtain $-\dfrac{t+1}{(t-1)^3}$. Very few realised that, given $t>0$, this expression can only be positive when the denominator is negative.

Question 6

(a) Almost all worked from one side of the equation to the other, after writing cosh and sinh in terms of exponentials, to fully justify the given identity.

(b) Almost all used the given substitution with the identity from **part (a)** and integrated to get $\dfrac{4}{3}\sinh^3 x+\dfrac{4}{5}\sinh^5 x+C$, although it was common to see the arbitrary constant omitted.

(c) Most candidates used the correct method to find the integrating factor, and, after multiplying both sides of the equation by their integrating factor, were able to attempt to integrate the RHS. Good candidates maintained accuracy throughout, particularly when applying their result from **part (b)** and substituting in the initial conditions. Most candidates remembered to add an arbitrary constant after integrating the right-hand side, substituted the initial conditions and made y the subject.

Question 7

(a) This part of the question was well done, though a few candidates accepted zero eigenvectors without checking for errors in their working. A few candidates spent time finding $\det(\mathbf{A}-\lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix.

(b) Good candidates were able to maintain accuracy when substituting $\mathbf{A}$ into the characteristic equation, both when calculating $\mathbf{A}^2$ and when making $\mathbf{A}^{-1}$ the subject.

Question 8

(a) Most candidates stated the formula for the sum of the geometric progression accurately. Some wrote the exponent as $n-1$, which hindered their progress in **part (b)**. A few incorrectly stated that the sum was equal to zero.

(b) Almost all knew that de Moivre's theorem related the series to the sum of the geometric progression in **part (a)**. The strongest candidates took the imaginary part, after multiplying the numerator and denominator by the conjugate of $z-1$ or by $z^{-\frac{1}{2}}$. Use of appropriate trigonometric identities then led to the given answer.

(c) Most candidates formed a correct expression for the sum of the areas of the rectangles. Good candidates emphasised the substitution of $\theta=\dfrac{1}{n}$ into the equation from **part (b)**, clearly showing the given upper bound.

(d) Strong candidates correctly adapted their solution to **part (c)** and gave a suitable lower bound.

FURTHER MATHEMATICS

Paper 9231/31
Further Mechanics 31

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagram as well.

Candidates are encouraged to pay particular attention in checking that the equations they write are dimensionally correct and consistent.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

Candidates are encouraged to draw a suitable diagram or, in cases where a diagram is provided, to annotate it, as this helps understand the problem and model it correctly. For example, in **Question 3** many candidates who annotated the diagram identified the best point about which to work out the equation of moments.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's 2nd Law, e.g. in questions involving collisions, they must ensure they explicitly mention the mass or masses involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of well-known formulae, as in **Question 5**.

Comments on specific questions

Question 1

(a) To answer this part question the candidates had to apply the Principle of Conservation of Linear Momentum. Many candidates did it correctly. A typical mistake was to consider that the velocity of particle A before and after the collision had the same direction.

(b) Many candidates attempted to use the information regarding the Kinetic Energy of particles A and B, however only the stronger candidates realised that particle A had both components, whereas particle B had only the vertical one. Most of the candidates who correctly set up the Kinetic Energy equation, obtained the correct answer, showing good algebraic manipulative skills.

(c) Many candidates attempted to apply Newton's Elastic Law, with various degrees of success. A typical mistake was in the signs of the velocities of approach.

Question 2

(a) In answering this question, many candidates correctly set up the differential equation and, after using the boundary condition, obtained the correct answer, showing a good understanding of the laws of logarithms. The expression for the integral of the velocities was available in the formula booklet, and stronger candidates used this fact to obtain a correct answer quickly.

(b) This part question was answered correctly by most of the candidates who scored well in **part (a)**. They showed a good understanding of how to apply limits.

Question 3

This question proved challenging for many candidates, who nevertheless often scored the marks allocated to the equilibrium of forces, including the use of the equation $F = \mu R$. Stronger candidates took moments about point P and quickly obtained the correct answer, often in an elegant way. In questions where many forces are involved candidates are encouraged to examine the diagram carefully before selecting the point about which to take moments. A good choice can greatly simplify the subsequent working, thus decreasing the chance of making errors as well as the time spent on calculations. The best answers included a fully annotated diagram, with an indication of the letters used to represent the forces active on the system. Weak candidates did not annotate the diagram and sometimes used letters that are not usually associated with such forces, e.g. X for the normal reaction, instead of N or R .

Question 4

(a) This part question asked the candidates to find the distance L between the particle P and one end of the string, and the candidates were expected to apply the Principle of Conservation of Mechanical Energy. Many candidates did so, although with varying degrees of success. In the most elegant answers, the energy equation was written $x = 2L - 8a$, instead of in terms of L . Weaker answers omitted one term of the Elastic Potential Energy.

(b) Only a few candidates answered this part question correctly. They almost all annotated the diagram provided in the question. The most common mistake was to use the equation $T = ma$ instead of $2t\cos(\theta) = ma$. Many candidates applied Hooke's law correctly to determine T .

Question 5

(a) This part question was answered correctly by most candidates. Weaker candidates derived the equation $y = x\,tan\alpha - \dfrac{gx^2}{2u^2}sec^2\alpha$ from $y = ut\,sin\alpha - \dfrac{1}{2}gt^2$ and $x = ut\,cos\alpha$, instead of using the formula available in the formula booklet.

(b) To answer this part question the candidates had to determine the range and time of flight of both particles and use the fact that the range was the same, whereas the ratio of the times of flight was $T_P : T_Q = 2:1$ to create two equations that would provide the two components of V . The last step was to use the components to obtain the answer. To answer this question the candidates used a variety of approaches with varying degrees of success. Most candidates derived the correct expressions for time of flight and range of particle P , but many of them did not manage to do the same for particle Q . A typical unsuccessful approach was to use the expression $\dfrac{v^2\sin 2\alpha}{g}$ for the range of particle Q .

Question 6

This question proved particularly challenging for candidates. Most of them used the equations for the equilibrium of forces in **part (a)**, then applied the Principle of Conservation of Energy in **part (b)** and, often, did not answer **part (c)**.

(a) To answer this part question the candidates had to realise that the angular velocity ω of the two particles is the same. Those who did, wrote the expressions for ω_P and ω_Q , equated them, and obtained the correct given answer.

(b) This part question required the candidates to consider the equilibrium of forces at P and Q , combine the two equations to eliminate the tension, and solve the resulting quadratic equation. Stronger candidates had no problems doing all these steps, often in a very neat manner. A common mistake among the candidates who applied the correct approach (equilibrium of forces) was in the signs of the tension and the component of weight along the rod. Weaker candidates attempted to use the Principle of Conservation of Mechanical Energy, without success.

(c) This last part question proved particularly challenging. Many candidates did not attempt to answer it. Those who did, often attempted to use the Principle of Conservation of Mechanical Energy. Even though they often did not include all the terms of the Kinetic Energy and Gravitational Potential Energy, they managed to write a dimensionally correct energy equation, therefore scoring at least one mark.

FURTHER MATHEMATICS

Paper 9231/32
Further Mechanics 32

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagram as well.

Candidates are encouraged to pay particular attention in checking that the equations they write are dimensionally correct and consistent.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

Candidates are encouraged to draw a suitable diagram or, in cases where a diagram is provided, to annotate it, as this helps understand the problem and model it correctly. For example, in **Question 7 parts (b)** and **(c)** many candidates who drew a diagram realised that the most efficient way to answer both part questions was to apply the equilibrium of forces in the direction perpendicular to the rod, rather than in the more conventional horizontal/vertical directions, and in most cases scored full marks.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's 2$^{\text{nd}}$ Law, e.g. in questions involving collisions, they must ensure they explicitly mention the mass or masses involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of well-known formulae, as in **Question 6 part (a)**.

Comments on specific questions

Question 1

Please note that due to a series-specific issue with question 1, full marks have been awarded to all candidates for this question to make sure that no candidates were disadvantaged.

Question 2

Please note that due to a series-specific issue with question 2, full marks have been awarded to all candidates for this question to make sure that no candidates were disadvantaged.

Question 3

(a) Many candidates successfully worked out the expressions for the areas of triangles ABC and BDE, the distances of their centres of mass from the line AC, and correctly used them in the moments equation to answer this part question. Stronger candidates simplified the answer into $\dfrac{a\left(k^2+k-2\right)}{3k\left(k+1\right)}$ and used the simplified expression to answer **part (b)**. Weaker candidates wrote the answer as the sum of three fractions $\frac{ak^2}{3k^2-3}-\frac{a}{k^2-1}+\frac{2ak^2}{3k^3-3k}$ and in most cases struggled to use their answer in **part (b)**. Some candidates did not write dimensionally consistent equations and scored no marks. A few candidates wrote dimensionally consistent but incorrect equations, e.g. they did not use the term a^2 in the expressions for the areas and, even though they obtained the correct expression for the answer, scored no marks, as the answer was obtained from an incorrect method.

(b) Most candidates correctly substituted their answer from **part (a)** into the formula $tan\theta=\frac{\bar{x}}{a}$. Stronger candidates rearranged the equation, solved it and correctly excluded all the solutions except the correct one, $k=3$, thus scoring full marks.

Question 4

(a) Many candidates applied Newton's Elastic Law and the Equation of Momentums correctly, even though a significant number of them did not include the mass in the latter equation, thus missing at least two marks. Stronger candidates solved the system of equations to work out the expression for v_A and then used it, along with the vertical component of the velocity of sphere B, to equate the times taken by the two spheres to hit the walls $\frac{d}{u\sin\theta}=\frac{d}{\frac{3}{4}u\cos\theta}$ and swiftly and elegantly obtained the correct answer.

(b) Most candidates included both components of the velocity of sphere B in the calculation of the final Kinetic Energy. Those who did it correctly, often scored full marks.

Question 5

(a) In answering this part question many candidates correctly applied the Principle of the Conservation of Energy, showing a good understanding of this topic. To answer this part question, they also had to write the equation for the equilibrium of forces at points A and B. While the former equation did not create problems to many candidates, only the stronger ones read the text carefully and realised that the reaction of the wire was in the direction OB (and not BO). Those who did, often correctly annotated the diagram, showed good algebraic manipulative skills and, almost always, obtained the correct answer.

(b) To answer this part question the candidates had to use their answer from **part (a)** to work out the value of v_A (or v_A^2) and get the answer by comparing it with the expression given in the text. Most candidates realised this and scored at least the method mark allocated for those steps.

Question 6

(a) This part question asked the candidates to derive the equation of the particle P in the form $y=x\,tan\alpha-\frac{gx^2}{2u^2}sec^2\alpha$. Most candidates realised that, as the answer was given, they had to show all the steps involved and many of them were able to do so, usually scoring full marks.

(b) In answering this part question, weaker candidates did not use the result given in **part (a)** but started from the equations $x=ut\cos\alpha, y=ut\sin\alpha-\frac{1}{2}gt^2$, often with poor outcomes. Stronger candidates expressed the equation given in **part (a)** in terms of $\tan\alpha$, substituted the values for u and x, eliminated h from the two equations, solved the quadratic equation obtained, and used the answers to calculate the values of h. Some candidates also calculated the values of α, even though this was not needed. Weaker candidates did not express the equations in terms of $\tan\alpha$ and only a handful of them managed to obtain the correct answer.

Question 7

(a) To answer this part question the candidates had to apply the Principle of the Conservation of Mechanical Energy to the system. Weaker candidates omitted the term representing the Gravitational Potential Energy. Many candidates showed a good understanding of the topic by obtaining the correct expressions for each component and, in most cases, the correct answer. A recurring mistake in working out the change in Elastic Potential Energy was to work out the square of the difference of the extensions, instead of the difference of the squares of the extensions.

(b) This part question, along with **part (c)**, proved challenging for most candidates. The best answers included an annotated diagrams of the forces involved. The strongest candidates realised that the most efficient way to answer both part questions was to apply the equilibrium of forces in the direction perpendicular to the rod, rather than in the more conventional horizontal/vertical directions. Those who did, often scored full marks, in some cases using a very elegant approach. Only a minority of the candidates who resolved the forces horizontally and vertically managed to score marks.

(c) As in **part (b)**, only the stronger candidates answered this part question correctly. The most elegant solutions used the fact that the tension in the rod immediately before it is released is $\frac{4}{3}$ of that on the string at the same time.

FURTHER MATHEMATICS

Paper 9231/33
Further Mechanics 33

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagram as well.

Candidates are encouraged to pay particular attention in checking that the equations they write are dimensionally correct and consistent.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

Candidates are encouraged to draw a suitable diagram or, in cases where a diagram is provided, to annotate it, as this helps understand the problem and model it correctly. For example, in **Question 3** many candidates who annotated the diagram identified the best point about which to work out the equation of moments.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's 2nd Law, e.g. in questions involving collisions, they must ensure they explicitly mention the mass or masses involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of well-known formulae, as in **Question 5**.

Comments on specific questions

Question 1

(a) To answer this part question the candidates had to apply the Principle of Conservation of Linear Momentum. Many candidates did it correctly. A typical mistake was to consider that the velocity of particle A before and after the collision had the same direction.

(b) Many candidates attempted to use the information regarding the Kinetic Energy of particles A and B, however only the stronger candidates realised that particle A had both components, whereas particle B had only the vertical one. Most of the candidates who correctly set up the Kinetic Energy equation, obtained the correct answer, showing good algebraic manipulative skills.

(c) Many candidates attempted to apply Newton's Elastic Law, with various degrees of success. A typical mistake was in the signs of the velocities of approach.

Question 2

(a) In answering this question, many candidates correctly set up the differential equation and, after using the boundary condition, obtained the correct answer, showing a good understanding of the laws of logarithms. The expression for the integral of the velocities was available in the formula booklet, and stronger candidates used this fact to obtain a correct answer quickly.

(b) This part question was answered correctly by most of the candidates who scored well in **part (a)**. They showed a good understanding of how to apply limits.

Question 3

This question proved challenging for many candidates, who nevertheless often scored the marks allocated to the equilibrium of forces, including the use of the equation $F = \mu R$. Stronger candidates took moments about point P and quickly obtained the correct answer, often in an elegant way. In questions where many forces are involved candidates are encouraged to examine the diagram carefully before selecting the point about which to take moments. A good choice can greatly simplify the subsequent working, thus decreasing the chance of making errors as well as the time spent on calculations. The best answers included a fully annotated diagram, with an indication of the letters used to represent the forces active on the system. Weak candidates did not annotate the diagram and sometimes used letters that are not usually associated with such forces, e.g. X for the normal reaction, instead of N or R.

Question 4

(a) This part question asked the candidates to find the distance L between the particle P and one end of the string, and the candidates were expected to apply the Principle of Conservation of Mechanical Energy. Many candidates did so, although with varying degrees of success. In the most elegant answers, the energy equation was written $x = 2L - 8a$, instead of in terms of L. Weaker answers omitted one term of the Elastic Potential Energy.

(b) Only a few candidates answered this part question correctly. They almost all annotated the diagram provided in the question. The most common mistake was to use the equation $T = ma$ instead of $2t\cos(\theta) = ma$. Many candidates applied Hooke's law correctly to determine T.

Question 5

(a) This part question was answered correctly by most candidates. Weaker candidates derived the equation $y = x\,tan\alpha - \dfrac{gx^2}{2u^2}sec^2\alpha$ from $y = ut\,sin\alpha - \dfrac{1}{2}gt^2$ and $x = ut\,cos\alpha$, instead of using the formula available in the formula booklet.

(b) To answer this part question the candidates had to determine the range and time of flight of both particles and use the fact that the range was the same, whereas the ratio of the times of flight was $T_P : T_Q = 2:1$ to create two equations that would provide the two components of V. The last step was to use the components to obtain the answer. To answer this question the candidates used a variety of approaches with varying degrees of success. Most candidates derived the correct expressions for time of flight and range of particle P, but many of them did not manage to do the same for particle Q. A typical unsuccessful approach was to use the expression $\dfrac{v^2 \sin 2\alpha}{g}$ for the range of particle Q.

Question 6

This question proved particularly challenging for candidates. Most of them used the equations for the equilibrium of forces in **part (a)**, then applied the Principle of Conservation of Energy in **part (b)** and, often, did not answer **part (c)**.

(a) To answer this part question the candidates had to realise that the angular velocity ω of the two particles is the same. Those who did, wrote the expressions for ω_P and ω_Q, equated them, and obtained the correct given answer.

(b) This part question required the candidates to consider the equilibrium of forces at P and Q, combine the two equations to eliminate the tension, and solve the resulting quadratic equation. Stronger candidates had no problems doing all these steps, often in a very neat manner. A common mistake among the candidates who applied the correct approach (equilibrium of forces) was in the signs of the tension and the component of weight along the rod. Weaker candidates attempted to use the Principle of Conservation of Mechanical Energy, without success.

(c) This last part question proved particularly challenging. Many candidates did not attempt to answer it. Those who did, often attempted to use the Principle of Conservation of Mechanical Energy. Even though they often did not include all the terms of the Kinetic Energy and Gravitational Potential Energy, they managed to write a dimensionally correct energy equation, therefore scoring at least one mark.

FURTHER MATHEMATICS

Paper 9231/41
Further Probability and Statistics 41

Key messages

In all questions candidates are advised to show all of their working as method is important as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working so that their solution is clear and complete.

When performing a hypothesis test, candidates should clearly state the result of the test in terms of accepting or rejecting the null hypothesis before giving their conclusion in context.

Care must be taken with the language used when interpreting the result of any test. A hypothesis test is not a proof, so some level of uncertainty is required when presenting a conclusion in context. In general, candidates should not suggest that acceptance of the null hypothesis provides evidence that the null hypothesis is true, but rather that there is insufficient evidence to reject it.

General comments

Almost all candidates attempted all of the questions. The standard was generally high with regard to test selection and computing test statistics, but candidates sometimes used the wrong tabular value or used the correct value incorrectly when drawing a conclusion.

The rubric for the paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates are therefore advised to use a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures throughout may result in an error in the final answer. This is particularly the case in questions where several different values are calculated, each depending on the previous one. Such rounding errors were sometimes seen in **Question 1**.

Comments on specific questions

Question 1

This question was answered well with most candidates successfully computing the correct statistic and comparing this with the correct tabular value. Sometimes there was confusion caused by the sign of the test statistic with comments such as '$-1.44 < 1.833$, accept H_0' seen. A common mistake was to have H_1: $\mu >$ 4.22 and sometimes this confusion was repeated in the conclusion for the test.

Question 2

This question was also answered well with most candidates recognising the need to perform a chi-squared test. The most common mistake was to have insufficient detail in the hypotheses, for example 'H_0: independent; H_1: dependent'. A small number of responses either had the hypotheses the wrong way round or omitted them entirely. Some candidates spoiled an otherwise perfect response by not including any level of uncertainty in their conclusion, with statements such as 'opinions and streets on which they live are independent' often seen.

Question 3

(a) Most candidates recognised the need to consider differences, but some were confused by the need to test for a reduction of at least 2 units. The null hypothesis 'H_0: $\mu_B - \mu_E = 0$' was frequently seen, sometimes alongside the correct alternative hypothesis. It was also common to see the wrong tabular value selected or even a z-value. A small number of candidates failed to recognise the need for a paired t-test and made little progress with the question.

(b) Only a minority of candidates gave a satisfactory answer to this part. The assumption is that the distribution of the population differences is normal. Most candidates gave an incomplete statement with the generic answer 'the distribution is normal' being the most common.

Question 4

(a) Candidates were asked to show that $m = \frac{1}{6}$ and to find the values of k and c. Since the value of m was given, candidates needed to provide sufficient detail in their working so that the derivation is clear and complete. Often this was present in terms of a definite integral or by referring to an appropriate area under the graph, but sometimes the value was obtained after simply stating that $2m = \frac{1}{3}$, which whilst correct is not sufficient to earn a mark. Finding the values of k and c required obtaining two independent equations in k and c. Most candidates derived at least one equation, usually by integrating and setting equal to $\frac{2}{3}$, but a smaller number of candidates were able to obtain the second equation by equating at $x = 2$.

(b) In this part, candidates were asked to find the exact interquartile range. Almost all candidates were able to find the correct value for Q_1. Candidates who did not find the correct values of k and c in **part (a)** were still able to obtain the two method marks for finding Q_3 and the final accuracy mark. A small number of candidates gave a numerical value rather than an exact value as required.

Question 5

This question was answered well by most candidates with many fully correct responses seen.

(a) This was a 'show that' question so candidates were required to include sufficient detail in their response so that the derivation of the probability generating function was clear. The best responses first derived $\text{P}(X = r)$ before writing the function in extensive form. They then clearly indicated that this was an infinite geometric progression, either by explicitly stating the first term and common ratio, or by using brackets in their expression to indicate its form, before applying the formula for the sum of an infinite GP.

(b) Most candidates knew what was required for this part and attempted to find the first and second derivatives of the probability function given in the stem of the question. The majority of candidates used the quotient rule to find the first derivative but often left the numerator unsimplified which sometimes led to errors in finding the second derivative. A number of candidates used the quotient rule again when finding the second derivative which was not necessary and often led to confusion and error. Most candidates knew how to use their derived functions to calculate the variance of X with only a small number of candidates leaving their expressions in terms of t. The most successful candidates substituted $t = 1$ into each derivative and simplified before combining to find $\text{Var}(X)$.

(c) A number of candidates omitted this part having struggled with **part (b)** but all that was required was to square the probability generating function given in the stem of the question after setting $p = \frac{1}{6}$.

Question 6

This question required an application of the Wilcoxon signed-rank test. Because of the size of the sample, a normal approximation was required. Many candidates used the correct approach to find the P and Q values from the data, but there were many accuracy errors. Candidates who drew a table with all the relevant values were usually more successful in working accurately than those who annotated the data in the question paper.

Only a minority of candidates stated suitable hypotheses. The most common errors were to use μ instead of m or to refer to the median of the data rather than population median.

The majority of candidates found the mean and variance of the normal distribution correctly though not all of them made further progress in finding the signed ranks. The test statistic was often found correctly, but a number of candidates omitted the continuity correction.

FURTHER MATHEMATICS

Paper 9231/42
Further Probability & Statistics 42

Key messages

In all questions candidates are advised to show all of their working as method is important as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working so that their solution is clear and complete.

When performing a hypothesis test, candidates should clearly state the result of the test in terms of accepting or rejecting the null hypothesis before giving their conclusion in context.

Care must be taken with the language used when interpreting the result of any test. A hypothesis test is not a proof so some level of uncertainty is required when presenting a conclusion in context. In general, candidates should not suggest that acceptance of the null hypothesis provides evidence that the null hypothesis is true, but rather that there is insufficient evidence to reject it.

General comments

Almost all candidates attempted all of the questions. The standard was generally high with regard to test selection and computing test statistics but candidates sometimes used the wrong tabular value or used the correct value incorrectly when drawing a conclusion.

The rubric for the paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates are therefore advised to use a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures throughout may result in an error in the final answer. This is particularly the case in questions where several different values are calculated, each depending on the previous one. Such rounding errors were often seen in **Question 1** and **Question 2(b)**.

Comments on specific questions

Question 1

The majority of candidates made good progress with this question with he majority giving their answer in the form [3.50, 4.96]. The more common errors were choosing the incorrect z-value or using a pooled estimate for the standard deviation. A very small number of candidates performed a hypothesis test instead of finding a confidence interval.

Question 2

(a) The vast majority of candidates obtained full marks.

(b) Almost all candidates made some progress with this question. For a fully correct solution, candidates needed to combine the first two columns and also the last two columns, so that the expected values are all at least 5. A number of candidates combined one pair of columns but not the other. Most candidates knew, and could apply, the correct formula for calculating the test statistic but the correct tabular value, consistent with their grouping, was not always chosen. The hypotheses were often stated with insufficient detail. It is expected that both the distribution and the data are mentioned in the hypotheses.

Question 3

This question was answered well by most candidates with many fully correct responses seen.

(a) A few candidates used sampling with replacement but most candidates used the correct method and obtained the correct probability generating function.

(b) This part was also answered well with the vast majority of candidates understanding that they needed to multiply their probability generating function obtained in **part (a)** with that given in **part (b)**. Very occasionally these functions were added instead of multiplied.

(c) Most candidates knew how to find the variance of *z* and, provided they had obtained the correct probability generating function in **part (b)**, went on to obtain full marks.

Question 4

Many candidates scored well on this question but sometimes used inefficient methods. The most concise solutions worked with the probability density function throughout. Candidates who were unable to obtain the correct values of *a* and *b* in **part (a)** were still able to obtain the method marks in **part (b)** and **part (c)**.

(a) Candidates were asked to show that $c = \frac{1}{8}$ and find the values of *a* and *b*. Since the value of *c* was given, candidates needed to provide sufficient detail in their working so that the derivation is clear and complete. Often this was present in terms of a definite integral or by referring to an appropriate area under the graph, but sometimes the value was obtained after simply stating that $2c = \frac{1}{4}$, which whilst correct is not sufficient to earn a mark. Finding the values of *a* and *b* required obtaining two independent equations in *a* and *b*. Most candidates derived at least one equation, usually by integrating and setting equal to $\frac{3}{4}$ or 1, but a smaller number of candidates were able to obtain the second equation by equating at $x = 4$.

(b) Most candidates formed a correct equation by setting $\int_0^m \frac{1}{128}\left(20x - x^3\right)dx = \frac{1}{2}$. This led to a quartic equation in *m* from which candidates were expected to pick out the appropriate value of $2\sqrt{2}$ for the median. A small number of candidates listed all four roots without indicating which one was the solution.

(c) The most successful candidates multiplied the probability density function given in the stem of the question by $\sqrt{x}$ and integrated. Those who had found the correct values of *a* and *b* in **part (a)** were usually successful here. A common alternative approach was to define a new random variable *Y* such that $\mathrm{E}(Y) = E\left(\sqrt{X}\right)$. Whilst this is a valid method it required more work and generated far fewer fully correct responses.

Question 5

(a) Only a minority of candidates stated suitable hypotheses. The most common errors were to use μ instead of *m* or to refer to the median of the data rather than the population median. Most candidates attempted to find a test statistic based on ranks but there were many accuracy errors. A number of candidates who correctly calculated 55 then chose $8(8+9+1) - 55 = 89$ as the test statistic. Candidates usually chose the correct tabular value but it was not uncommon to see the correct statistic and tabular value followed by incorrectly rejecting the null hypothesis. A number of candidates, having correctly accepted the null hypothesis, concluded that this supported the manager's claim.

(b) Please note that due to a series-specific issue with **part 5(b)**, full marks have been awarded to all candidates for this part to make sure that no candidates were disadvantaged.

(c) Only a few candidates provided sufficient detail to earn the mark for this part. The most common responses stated the different assumptions required for each test rather than why they might give different results. The most common correct answer made reference to the fact that the existence of outliers has a greater effect on a *t*-test than on a Wilcoxon signed-rank test.

FURTHER MATHEMATICS

Paper 9231/43
Further Probability and Statistics 43

Key messages

In all questions candidates are advised to show all of their working as method is important as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working so that their solution is clear and complete.

When performing a hypothesis test, candidates should clearly state the result of the test in terms of accepting or rejecting the null hypothesis before giving their conclusion in context.

Care must be taken with the language used when interpreting the result of any test. A hypothesis test is not a proof, so some level of uncertainty is required when presenting a conclusion in context. In general, candidates should not suggest that acceptance of the null hypothesis provides evidence that the null hypothesis is true, but rather that there is insufficient evidence to reject it.

General comments

Almost all candidates attempted all of the questions. The standard was generally high with regard to test selection and computing test statistics, but candidates sometimes used the wrong tabular value or used the correct value incorrectly when drawing a conclusion.

The rubric for the paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates are therefore advised to use a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures throughout may result in an error in the final answer. This is particularly the case in questions where several different values are calculated, each depending on the previous one. Such rounding errors were sometimes seen in **Question 1**.

Comments on specific questions

Question 1

This question was answered well with most candidates successfully computing the correct statistic and comparing this with the correct tabular value. Sometimes there was confusion caused by the sign of the test statistic with comments such as '$-1.44 < 1.833$, accept H_0' seen. A common mistake was to have H_1: $\mu > 4.22$ and sometimes this confusion was repeated in the conclusion for the test.

Question 2

This question was also answered well with most candidates recognising the need to perform a chi-squared test. The most common mistake was to have insufficient detail in the hypotheses, for example 'H_0: independent; H_1: dependent'. A small number of responses either had the hypotheses the wrong way round or omitted them entirely. Some candidates spoiled an otherwise perfect response by not including any level of uncertainty in their conclusion, with statements such as 'opinions and streets on which they live are independent' often seen.

Question 3

(a) Most candidates recognised the need to consider differences, but some were confused by the need to test for a reduction of at least 2 units. The null hypothesis 'H_0: $\mu_B - \mu_E = 0$' was frequently seen, sometimes alongside the correct alternative hypothesis. It was also common to see the wrong tabular value selected or even a z-value. A small number of candidates failed to recognise the need for a paired t-test and made little progress with the question.

(b) Only a minority of candidates gave a satisfactory answer to this part. The assumption is that the distribution of the population differences is normal. Most candidates gave an incomplete statement with the generic answer 'the distribution is normal' being the most common.

Question 4

(a) Candidates were asked to show that $m = \frac{1}{6}$ and to find the values of k and c. Since the value of m was given, candidates needed to provide sufficient detail in their working so that the derivation is clear and complete. Often this was present in terms of a definite integral or by referring to an appropriate area under the graph, but sometimes the value was obtained after simply stating that $2m = \frac{1}{3}$, which whilst correct is not sufficient to earn a mark. Finding the values of k and c required obtaining two independent equations in k and c. Most candidates derived at least one equation, usually by integrating and setting equal to $\frac{2}{3}$, but a smaller number of candidates were able to obtain the second equation by equating at $x = 2$.

(b) In this part, candidates were asked to find the exact interquartile range. Almost all candidates were able to find the correct value for Q_1. Candidates who did not find the correct values of k and c in **part (a)** were still able to obtain the two method marks for finding Q_3 and the final accuracy mark. A small number of candidates gave a numerical value rather than an exact value as required.

Question 5

This question was answered well by most candidates with many fully correct responses seen.

(a) This was a 'show that' question so candidates were required to include sufficient detail in their response so that the derivation of the probability generating function was clear. The best responses first derived P $(X = r)$ before writing the function in extensive form. They then clearly indicated that this was an infinite geometric progression, either by explicitly stating the first term and common ratio, or by using brackets in their expression to indicate its form, before applying the formula for the sum of an infinite GP.

(b) Most candidates knew what was required for this part and attempted to find the first and second derivatives of the probability function given in the stem of the question. The majority of candidates used the quotient rule to find the first derivative but often left the numerator unsimplified which sometimes led to errors in finding the second derivative. A number of candidates used the quotient rule again when finding the second derivative which was not necessary and often led to confusion and error. Most candidates knew how to use their derived functions to calculate the variance of X with only a small number of candidates leaving their expressions in terms of t. The most successful candidates substituted $t = 1$ into each derivative and simplified before combining to find Var (X).

(c) A number of candidates omitted this part having struggled with **part (b)** but all that was required was to square the probability generating function given in the stem of the question after setting $p = \frac{1}{6}$.

Question 6

This question required an application of the Wilcoxon signed-rank test. Because of the size of the sample, a normal approximation was required. Many candidates used the correct approach to find the P and Q values from the data, but there were many accuracy errors. Candidates who drew a table with all the relevant values were usually more successful in working accurately than those who annotated the data in the question paper.

Only a minority of candidates stated suitable hypotheses. The most common errors were to use μ instead of m or to refer to the median of the data rather than population median.

The majority of candidates found the mean and variance of the normal distribution correctly though not all of them made further progress in finding the signed ranks. The test statistic was often found correctly, but a number of candidates omitted the continuity correction.